TOWARD A RING SPECTRUM MAP FROM $K(ku)$ TO E_2

BY

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THESIS FOR THE DEGREE OF

MASTER OF SCIENCE

(MASTER I MATEMATIKK)

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FACULTY OF MATHEMATICS AND NATURAL SCIENCES
UNIVERSITY OF OSLO

MAY 2015

Abstract

In this thesis we compute the K_2-homology of $B^3\mathbf{Z} = K(\mathbf{Z}, 3)$ following Ravenel-Wilson, and exhibit all ring spectrum maps from $\Sigma^\infty B^3\mathbf{Z}_+$ to K_2. We define the complex oriented theory K_n as a quotient of Lubin-Tate theory E_n (related to Morava K-theory $K(n)$ and Johnson-Wilson theory $E(n)$, respectively), and provide background from stable homotopy theory as well as the algebra necessary for Ravenel-Wilson's computation. This computation is a stepping-stone in a program to produce a ring spectrum map from $K(ku)$ to E_2.

Contents

Introduction

The algebraic K-theory of number fields, and of their rings of integers, is currently best understood in terms of Galois and étale descent. A number field $\mathbf{F}$ is viewed as the fixed field of its absolute Galois group $G_{\mathbf{F}}$, acting on the field $\overline{\mathbf{F}} = \overline{\mathbf{Q}}$ of algebraic numbers. Its algebraic K-theory $K(\mathbf{F})$ is well approximated by the homotopy fixed points of $G_{\mathbf{F}}$ acting on $K(\overline{\mathbf{F}}) = K(\overline{\mathbf{Q}})$, which is known to be p-adically equivalent to connective complex K-theory, ku, by way of the embedding $\overline{\mathbf{Q}} \to \overline{\mathbf{C}}$.

The p-adic complex K-theory spectrum, $ku\widehat{_p}$, is also known as the first Lubin-Tate spectrum E_1. Calculations by Ausoni-Rognes [AR02] suggest that $K(ku)$ may also be well approximated by homotopy fixed points of an action on the second Lubin-Tate spectrum E_2. There is no known analogue of the embedding $\mathbf{F} \to \overline{\mathbf{F}} = \overline{\mathbf{Q}} \subset \mathbf{C}$ in this setting, so even the construction of a map $K(ku) \to E_2$ that would correspond to the canonical map from the homotopy fixed points remains an open problem.

The algebraic K-theory space of ku is a group completion of the topological monoid

$$M = \coprod_{k>0} BGL_k(ku).$$

In other words $\Omega^\infty K(ku) = \Omega BM$. There is an inclusion $BGL_1(ku) \to \Omega BM$, and inside $BGL_1(ku)$ sits a copy of $B^3\mathbf{Z}$. If $B^3\mathbf{Z}_+ \to E_2$ is a ring spectrum map, it is hoped that this extend successively to $BGL_1(ku)$ and all of BM, to finally yield a ring spectrum map $K(ku) \to E_2$. Given a map $B^3\mathbf{Z}_+ \to K_2$ we may lift this map to $B^3\mathbf{Z}_+ \to E_2$ and proceed with this program.

Our main result, Proposition 7.0.6, is that the ring spectrum maps $\Sigma^\infty B^3\mathbf{Z}_+ \to K_2$ are in one-to-one correspondence with characters

$$K_*B^3\mathbf{Z} \cong (K_*B^3\mathbf{Z})^* \to \mathbf{F}_p^*.$$

Note that

$$K_*B^3\mathbf{Z} \cong \operatorname{colim}_j B^2\mathbf{Z}/(p^j) \cong \bigotimes_{k\geq 0} K_*[b_{(2k,1)}]/(b_{(2k,1)}^p + b_{(2k,1)}).$$

These characters are determined by the images of $b_{(2k,1)}$ in $\mathbf{F}_p^*$ for $k = 0, 1, 2, \ldots$, and as such there are uncountably many. By Lemma 7.2.2 they depend on solving the equation

$$x^{p-1} + 1 = 0,$$

and so there are no non-trivial ring spectrum maps from $\Sigma^\infty B^3\mathbf{Z}_+$ to the second Morava K-theory $K(2)$, contrary to what was first expected.

Organization of chapters. This thesis is divided into three parts.

Part 1, Chapters 1 through 4, is a review of background material from stable homotopy theory and algebra. In Chapter 1 we develop the closed symmetric monoidal model category of symmetric spectra, which is a model for the stable homotopy category. In Chapter 2 we sketch the construction a symmetric ring spectrum model for connective complex K-theory, ku, and present the algebraic K-theory spectrum of a (commutative) ring spectrum via the Waldhausen construction. In Chapter 3 we develop the basic notions of complex oriented spectra, formal group laws, and exhibit a model for complex bordism MU. We then define the Lubin-Tate spectra E_n using the Landweber exact functor theorem. The spectrum K_n is then a quotient of E_n. In Chapter 4 we develop the theory of Tor and Ext necessary for the computations of Chapter 6.

Part 2, Chapters 5 and 6, is an adaption of Ravenel-Wilson's computations for the space $B^3\mathbf{Z}$ using K_2 in place of $K(2)$. In Chapter 5 we develop the bar spectral sequence with its Hopf algebra structure, which is necessary for Chapter 6, and provide as example a thorough computation of $H_*(B^3\mathbf{Z}; \mathbf{F}_p)$ (up to multiplicative extensions). In Chapter 6 we then compute the Hopf algebra $K_{2*}B^3\mathbf{Z}$.

Part 3, Chapter 7, is a simple but apparently novel computation of the set of ring spectrum maps $\Sigma^\infty B^3\mathbf{Z}_+ \to K_2$.

Acknowledgements First of all I would like to thank my advisor, John Rognes. For his incredible patience and kindness, and for his willingness to share his expertise in seeing me through my years of study at the University of Oslo. It has been way beyond his call of duty, and is deeply appreciated. I would also like to thank Kristian J. Moi for waking my interest in mathematics, and for helping me keep interested through innumerable conversations, musings and late, late evenings. His comments and suggestions on drafts of this thesis were of more help than I could ever have asked for. I would like to thank Irakli Patchkoria for giving me opportunity to speak at the University of Copenhagen. His encouragement and comments on the scope of this thesis' topic helped calm my nerves about what could be realistically accomplished. I would also like to thank Bjørn I. Dundas for inviting me to speak at the University of Bergen, and for encouraging me to apply for a PhD. I am also grateful to Knut Berg and Sigurd Segtnan, who read through parts of this manuscript and caught many, many mistakes. The retreats I had with my dear sister Mathilde S. Dahl, and my friend Nikolai Fullman were instrumental in finding peace to write anything at all. I would also like to thank Simen Lønsethagen for helpful discussions on model categories. Finally, I would also like to thank my parents and my friends for their understanding and support, without which any work would ever hold much value to me. This thesis is dedicated to the memory of my late grandfather Birger H. Dahl whose curiosity and playfulness in exploring topics he did not yet understand, I very much admired.

1 Notes on stable homotopy theory

In this section we give notes on some standard background to stable homotopy theory which typically falls outside the scope of a first exposure to algebraic topology. First some motivation. Recall that singular cohomology is represented in each degree n, up to homotopy, by an Eilenberg-Mac Lane space $K(G,n)$. In other words, there is a natural isomorphism $H^n(X;G) \cong [X_+, K(G,n)]$. The suspension isomorphism in cohomology is realized on the level of spaces by maps $\Sigma K(G,n) \to K(G,n+1)$. Collating the data

$$\Sigma K(G,n) \to K(G,n+1)$$

for each $n \geq 0$ we arrive at the notion of a spectrum. A *sequential spectrum* is a sequence X_n of pointed spaces, $n \geq 0$, together with structure maps $\Sigma X_n \to X_{n+1}$. Morphisms of spectra are sequences of maps $f_n : X_n \to Y_n$ compatible with the structure maps. Generalized cohomology theories are in general represented (up to homotopy) by spectra. There is a notion of weak equivalence between spectra, and the theory of model categories turns out to be a good framework for constructing a homotopy category of spectra. Given two finite pointed CW spaces X and Y the sequence of functions

$$\Sigma : [X,Y] \to [\Sigma X, \Sigma Y] \to [\Sigma^2 X, \Sigma^2 Y] \to \cdots$$

eventually stabilizes, and the colimit $\{X,Y\}$ is the set of *stable homotopy classes* of maps $X \to Y$. Stable homotopy equivalences are maps that have homotopy inverses after n-fold suspension for some n. Spectra as representing objects for generalized cohomology theories are blind to all but the stable maps $\{X,Y\}$. In other words, through the eyes of spectra two spaces are equivalent if there is a stable equivalence $X \to Y$. Dually, two spectra should be considered equivalent if they induce the same cohomology theory on the category of finite pointed CW spaces and stable maps, after Spanier-Whitehead [SW53]. To accomplish this we first induce a model structure on spectra from the category of spaces, and then modify it via Bousfield localization to make Σ an equivalence of homotopy categories.

In order to produce good representing objects for multiplicative cohomology theories we want to define (commutative) monoid objects in spectra. This requires a (symmetric) monoidal structure on the category of spectra. The naïve construction of the category of spectra above does not carry a symmetric monoidal structure, but many modern constructions do. The category of symmetric spectra is a monoidal model category, and as such has a closed symmetric monoidal structure compatible with the model structure. This ensures that its homotopy category also has a closed symmetric monoidal structure. Further, a 'monoid axiom' ensures the existence of a model structure on associative monoids in the category and on categories of modules over these. In order to induce a model structure on commutative monoids, some care must be taken with the model structure on spectra, leading to positive model structures. Though not strictly needed for our purposes, this is developed in the final subsection.

1.1 Model categories

Given a category with a subcategory of 'weak equivalences' one may hope that a localization of the category with respect to the weak equivalences, in much the same way one inverts the elements of a multiplicatively closed subset in a ring, produces a reasonable 'homotopy category,' as a generalization of the category of CW spaces and weak equivalences, and its homotopy category of CW spaces. However, with no further control the class of morphisms between two objects in such a localization does not generally form a set. Model categories, which were developed by Quillen [Qui67] adress this issue by introducing two more classes of maps, cofibrations and fibrations. In any model category there is a notion of cylinder and path object. The morphisms between two objects in the localization of a model category with respect to weak equivalences are then in bijection with a quotient of a set of maps between two closely related objects, by a notion of homotopy, which we take to be the homotopy classes of maps. We recall here the definition of a model category following Hovey [Hov99]. For further details on model categories and standard results, see Quillen [Qui67] or [Hov99].

We begin with two definitions that figure in the definition of a model category. For the rest of this subsection let $\mathscr{C}$ be a category.

Definition 1.1.1. Let f and g be morphisms in $\mathscr{C}$. We say that f is a *retract* of g if there are maps q, r, q', r' such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "q'"] \arrow[d, "f"'] & Z \arrow[r, "r'"] \arrow[d, "g"] & A \arrow[d, "f"] \\ X \arrow[r, "q"] & Y \arrow[r, "r"] & X, \end{tikzcd}$$

with $r \circ q = \mathrm{id}_X$ and $r' \circ q' = \mathrm{id}_A$.

Definition 1.1.2. Let f and g be morphisms in $\mathscr{C}$. We say that f has the left lifting property with respect to g, and that g has the right lifting property with respect to f if for any square diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & Z \arrow[d, "g"] \\ X \arrow[r] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

there exists a lift h making the diagram commute.

Definition 1.1.3. Let $\mathscr{C}$ be a category with all small limits and colimits, and with a triple of subcategories $\mathscr{M} = (\mathscr{W}, \mathscr{C}_c, \mathscr{C}_f)$, categories whose morphisms are called weak equivalences, cofibrations and fibrations, respectively. Call a map a *trivial cofibration* (resp. *trivial fibration*) if it is simultaneously a weak equivalence and a cofibration (resp. *fibration*). We say that $\mathscr{C}$ is a *model category* if

1. If f and g are two composable morphisms for which two of f, g and gf are weak equivalences, then so is the third.
2. If f and g are morphisms in $\mathscr{C}$ with f a retract of g, and g is a weak equivalence, cofibration, or fibration, then so is f.
3. Cofibrations have the left lifting property with respect to trivial fibrations, and fibrations have the right lifting property with respect to trivial cofibrations.
4. Any morphism $f : X \to Y$ factors functorially as both a trivial cofibration followed by a fibration, and a cofibration followed by a trivial fibration.

A choice of functorial factorization is taken to be part of the structure.

Remark. From the erreta to [Hov99], a *functorial factorization* in a category $\mathscr{C}$ is a pair (α, β) of functors $\mathrm{Ar}\,\mathscr{C} \to \mathrm{Ar}\,\mathscr{C}$ of the arrow category such that

- $\mathrm{dom} \circ \alpha = \mathrm{dom}$,
- $\mathrm{codom} \circ \alpha = \mathrm{dom} \circ \beta$,
- $\mathrm{codom} \circ \beta = \mathrm{codom}$, and
- $f = \beta(f) \circ \alpha(f)$ for all maps $f : X \to Y$ in $\mathscr{C}$.

Denote by $\emptyset$ the initial object in a model category $\mathscr{C}$, and $*$ the terminal object. An object X in $\mathscr{C}$ is called *cofibrant* if $\emptyset \to X$ is a cofibration, and *fibrant* if $X \to *$ is a fibration. By factoring the unique morphism $\emptyset \to X$ into a cofibration followed by a trivial fibration, as

$$\emptyset \to QX \to X$$

we see that every object is weakly equivalent to a cofibrant object, a *cofibrant replacement* QX of X. By factoring the unique map $X \to *$ into a trivial cofibration followed by a fibration, as

$$X \to PX \to *$$

we get a *fibrant replacement* PX of X. Using the functorial factorizations of morphisms in the model category, these replacements may be taken to be functorial in X.

Definition 1.1.4. If $\mathscr{C}$ is a model category, then the *homotopy category* of $\mathscr{C}$ is $\mathrm{Ho}(\mathscr{C}) = \mathscr{C}[\mathscr{W}^{-1}]$, where $\mathscr{W}$ is the category of weak equivalences in $\mathscr{C}$. The category $\mathscr{C}[\mathscr{W}^{-1}]$, which is constructed in [Qui67], is initial among categories $\mathscr{H}$ with a functor $\mathscr{C} \to \mathscr{H}$ sending morphisms in $\mathscr{W}$ to isomorphisms.

Remark. There is a notion of homotopy of maps from a cofibrant object to a fibrant object in any model category $\mathscr{C}$, and defining a new category with morphism sets to be $[X, Y] = \mathscr{C}(QX, PY)/\sim$, the set of maps in $\mathscr{C}$ between a cofibrant replacement of X to a fibrant replacement of Y modulo the relation of homotopy, we arrive at a more classical definition of homotopy category, shown in [Qui67] to be equivalent to $\mathscr{C}[\mathscr{W}^{-1}]$.

Example 1.1.5. The *Quillen model structure*, or *q-model structure*, on the category $\mathscr{T}$ of (compactly generated weak Hausdorff) spaces has fibrations the Serre fibrations, and weak equivalences maps $f : X \to Y$ which are π_0-bijections, and π_*-isomorphisms for $* > 0$ and every choice of basepoint. Cofibrations are retracts of 'relative cell complexes'. We explore this model structure more closely below.

A morphism of model categories should induce a morphism of homotopy categories, while two model categories should be considered equivalent if a morphism between them induces an equivalence of homotopy categories. This is captured by the following definition.

Definition 1.1.6. Let $\mathscr{C}$ and $\mathscr{D}$ be model categories and $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ two functors with F left adjoint to G. We say that the adjunction $F \dashv G$ is a *Quillen adjunction* if F is a *left Quillen functor*, that is, F is a left adjoint and preserves cofibrations and trivial cofibrations, and G is a *right Quillen functor*, meaning it preserves fibrations and trivial fibrations.

Remark. If F is a left Quillen functor then its right adjoint is automatically a right Quillen functor.

Definition 1.1.7. Let $F : \mathscr{C} \to \mathscr{D}$ be a left Quillen functor with right adjoint $G : \mathscr{D} \to \mathscr{C}$. Then the *left* (resp. *right*) *derived functor* of F, $LF : Ho(\mathscr{C}) \to Ho(\mathscr{D})$ (resp. $RF : Ho(\mathscr{D}) \to Ho(\mathscr{C})$) is defined by taking $LF(X) = F(QX)$ (resp. $RF(X) = F(PX)$), where QX (resp. PX) is a cofibrant (resp. fibrant) replacement of X. The adjunction $F \dashv G$ is a *Quillen equivalence* if LF or RG is an equivalence of categories.

The usual way to specify a model structure is to specify a class of cofibrations and trivial cofibrations that generate the model structure. This leads to the notion of a cofibrantly generated category. We first provide two definitions. Let I be a class of maps in a category $\mathscr{C}$. Then

- I-inj is the class of maps in $\mathscr{C}$ with the right lifting property with respect to maps of I.

- I-cell, the *relative cell complexes in* $\mathscr{C}$, is the class of transfinite compositions in $\mathscr{C}$ of pushouts with elements in I.

The relative cell complexes in $\mathscr{C}$ are essentially generalized relative CW spaces.

Definition 1.1.8 (Cofibrantly generated model category)**.** Let $\mathscr{C}$ be a model category. Then $\mathscr{C}$ is *cofibrantly generated* [Hov99] if there are sets of maps I and J such that

- the domains of maps in I are small relative to I-cell, in the sense of Appendix A,

- the domains of maps of J are small relative to J-cell,

- the class of fibrations is J-inj, and

- the class of trivial fibrations is I-inj.

We call I the set of *generating cofibrations*, and J the set of *generating trivial cofibrations*.

Example 1.1.9 (The Quillen model structure)**.** Let again $\mathscr{T}$ be the category of (compactly generated weak Hausdorff) topological spaces, and denote by Δ^n the topological n-simplex. Let I be the set of inclusions $\partial\Delta^n \to \Delta^n$, and J the set of inclusions of the horns $\Lambda^n_k \to \Delta^n$, $0 < n$, $0 \leq k \leq n$. Then I and J generate a model structure on $\mathscr{T}$. Existence of a functorial factorization follows from the small object argument, recorded for convenience in Appendix A.3. The Quillen model structure is also called the q-model structure. In this case the homotopy category is the usual homotopy category of CW spaces.

Example 1.1.10. With $\mathscr{T}$ as above, there is a second common model structure on $\mathscr{T}$ called the Hurewicz or h-model structure, or the Strøm model structure after Strøm [Str72]. Its cofibrations, fibrations and weak equivalences are closed Hurewicz cofibrations, Hurewicz fibrations, and homotopy equivalences.

Lemma 1.1.11 (Lemma 2.1.20 [Hov99])**.** *If $(F, G, \phi) : \mathscr{C} \to \mathscr{D}$ is an adjunction between model categories and $\mathscr{C}$ cofibrantly generated by I and J, then (F, G, ϕ) is a Quillen adjunction if and only if F takes generating cofibrations to cofibrations, and generating trivial cofibrations to trivial cofibrations.*

Example 1.1.12. Let $s\mathscr{E} = \mathscr{E}^{\Delta^{op}}$ be the category of simplicial sets. Geometric realization and the singular set functor $\mathrm{Sing}_.$ form an adjunction

$$\mathscr{T}(|X|, Y) \cong s\mathscr{E}(X, \mathrm{Sing}_. Y),$$

see [GJ09, Chapter I, Proposition 2.2]. This induces a model structure on $s\mathscr{E}$. Its fibrations are Kan fibrations, and its weak equivalences are q-equivalences after geometric realization.

Example 1.1.13. Given a small category I and a model category $\mathscr{C}$, the category $\mathscr{C}^I$ if I-shaped diagrams in $\mathscr{C}$ inherits a *projective* model structure by taking the cofibrations and equivalences to be component-wise cofibrations and weak equivalences, with fibrations defined by having the right lifting property with respect to trivial cofibrations.

1.2 Bousfield localization

Given a model category $\mathscr{C}$ with weak equivalences $\mathscr{W}$, and a subcategory $\mathscr{W}'$ of $\mathscr{C}$ containing W, it is under certain conditions on $\mathscr{W}'$ and $\mathscr{C}$ possible to modify the model structure on $\mathscr{C}$ to have maps of $\mathscr{W}'$ as weak equivalences. This is done systematically in the theory of *Bousfield localization* of model structures. Bousfield localization comes in a left and right variant where the cofibrations and fibrations, respectively, remain unchanged. We exhibit here the left version. The standard reference for Bousfield localization is [Hir03].

Definition 1.2.1. A category $\mathscr{C}$ is a *simplicial category* if it is enriched, tensored and cotensored in $s\mathscr{E}$, see [GJ09, Chapter 2, Section 2] for details.

Notation. We will write $\mathrm{map}(X, Y)$ for the simplicial set of maps $X \to Y$.

Definition 1.2.2. If $\mathscr{C}$ is also a model category we say that $\mathscr{C}$ is a *simplicial model category* if for every cofibration $f : Y \to X$ and fibration $g : T \to U$, the canonical map of simplicial sets

$$\mathrm{map}(X, T) \to \mathrm{map}(Y, T) \times_{\mathrm{map}(Y,U)} \mathrm{map}(X, U)$$

is a fibration of simplicial sets, trivial whenever f or g is.

Definition 1.2.3. Let $\mathscr{C}$ be a simplicial model category and S a class of maps in $\mathscr{C}$ with cofibrant domains. An object X in $\mathscr{C}$ is *S-local* if for every $f : A \to B \in S$ the induced map

$$\mathrm{map}(B, PX) \to \mathrm{map}(A, PX)$$

of simplicial sets is a weak equivalence, where PX is a fibrant replacement of X. A cofibration $f : A \to B$ is an *S-local weak equivalence* if for each S-local object X, $\mathrm{map}(QB, PX) \to \mathrm{map}(QA, PX)$ is a weak equivalence.

Definition 1.2.4. Let S be a class of maps in a simplicial model category $\mathscr{C}$ with model structure $\mathscr{M} = (\mathscr{W}, \mathscr{C}_c, \mathscr{C}_f)$. Then, if it exists, the *left Bousfield localization* of $\mathscr{C}$ with respect to S, written $L_S\mathscr{C}$, is the category $\mathscr{C}$ equipped with the model structure $L_S\mathscr{M} = (W_S, \mathscr{C}_c, \mathscr{C}_{f,S})$, where W_S is the category whose morphisms are S-local weak equivalences in $\mathscr{M}$, $\mathscr{C}_c$ is the category of cofibrations in $\mathscr{M}$, and morphisms in $\mathscr{C}_{f,S}$ have the right lifting property with respect to S-local weak equivalences in $\mathscr{C}_c$.

The following properties figure into the hypotheses of Theorem 1.2.7 giving existence of left Bousfield localizations of certain simplicial model category.

Definition 1.2.5. Let $\mathscr{C}$ be a model category. Then $\mathscr{C}$ is *left proper* if the pushout of a weak equivalence along a cofibration is a weak equivalence.

Recall that a morphism $f : X \to Y$ in a category with pushouts is an *effective monomorphism* if the diagram

$$X \xrightarrow{f} Y \rightrightarrows Y \cup_X Y,$$

with the canonical pair of morphisms $Y \to Y \cup_X Y$, is an equalizer.

Definition 1.2.6. Let $\mathscr{C}$ be a cofibrantly generated model category with generating cofibrations I and generating trivial cofibrations J. Then $\mathscr{C}$ is *cellular* if

- the domains and codomains of elements of I are compact relative to I, in the sense of Appendix A,
- the domains of elements of J are small relative to I, and
- all cofibrations are effective monomorphisms.

Theorem 1.2.7 (Section 3 [Hir03])**.** *Let $\mathscr{C}$ be a simplicial, left proper, cofibrantly generated and cellular model category, and let S be a (small) set of maps in $\mathscr{C}$. Then the Bousfield localization $L_S\mathscr{C}$ exists and is itself a simplicial, left proper, cofibrantly generated and cellular model category. The fibrant objects of $L_S\mathscr{C}$ are precisely the S-local objects.*

1.3 Monoidal model categories and the monoid axiom

A monoidal model category is a closed symmetric monoidal category with a model structure that induces a closed symmetric monoidal structure on its homotopy category [SS00], see Appendix B.2. For a monoidal model category to induce a model structure on the category of associative monoids in the category, and on on categories of modules over these, it id sufficient that the category satisfy a 'monoid axiom' of Schwede-Shipley [SS00]. To get a model structure on the category of commutative monoids requires a positive model structure, an issue solved for very general spectra (those in [Hov01]) by Gorchinskiy-Guletskii [GG15], for $\mathscr{I}$-spaces (and Γ-spaces) in [SS12], and diagram spectra (such as symmetric spectra) as well as Γ-spaces in [Man+01].

Let $f : X \to Y$ and $g : Z \to W$ be two maps in a monoidal category $\mathscr{C}$. Then the *pushout product* of f and g, $f\Box g$, is given by the induced map

$$f\Box g : Y \otimes Z \coprod_{X\otimes Z} X \otimes W \to Y \otimes W.$$

Definition 1.3.1. Let $\mathscr{C}$ be a symmetric monoidal category with model structure $\mathscr{M}$ and unit S, and let $QS \to S$ be a cofibrant replacement of S. Then $\mathscr{C}$ is a (*symmetric*) *monoidal model category* if

- for every pair of cofibrations f, g in $\mathscr{C}$ the pushout product $f\Box g$ is a cofibration in $\mathscr{C}$ such that if f or g is trivial, then $f\Box g$ is also trivial, and
- for all cofibrant objects X the map $X \otimes QS \to X \otimes S \cong X$ is a weak equivalence.

The following is [SS00, Definition 3.3].

Definition 1.3.2 (Monoid axiom)**.** A monoidal model category $\mathscr{C}$ satisfies the *monoid axiom* if every map in $(\{\text{acyclic cofibration}\})$-cell is a weak equivalence.

Theorem 1.3.3 (Theorem 4.1 (2), [SS00])**.** *Let $\mathscr{C}$ be a cofibrantly generated monoidal model category satisfying the monoid axiom, for which every object is small relative to the whole category, see Appendix A. Then if R is a commutative monoid in $\mathscr{C}$, the category of modules over R is a cofibrantly generated monoidal model category satisfying the monoid axiom R.*

1.4 Symmetric spectra

A problem arises in wanting to represent the multiplicative structure of multiplicative cohomology theories on the level of sequences of spaces: no good construction of a smash product (a closed symmetric monoidal structure) on the category of sequential spectra exists. Symmetric spectra is one of many modern constructions of a category of spectra with smash products and, with a suitable model structure, its homotopy category is equivalent to the usual stable homotopy category of [BF78].

The standard source on symmetric spectra is Hovey-Shipley-Smith [HSS00], while a more comprehensive textbook is in progress due to Schewde [Sch]. A general construction of symmetric spectra in general monoidal model categories is due to Hovey [Hov01]. With these newfound multiplicative models for spectra it is desirable to speak of ring spectra and categories of modules over them in a homotopy invariant way.

Denote by Σ the category whose objects are the finite sets $\mathbf{0} := \emptyset$ and $\mathbf{n} = \{1, \dots, n\}$, $n > 0$, with morphisms

$$\Sigma(\mathbf{m}, \mathbf{n}) = \begin{cases} \Sigma_n & \text{if } m = n \\ \emptyset & \text{otherwise.} \end{cases}$$

Let $\mathscr{T}_*$ be the category of pointed spaces, and let $\mathscr{T}_*^\Sigma$ be the category of functors $\Sigma \to \mathscr{T}_*$ and natural transformations. The objects are sequences of spaces X_n with Σ_n-action for $n = 0, 1, \dots$; these are the *symmetric sequences* in $\mathscr{T}_*$. Suppose that $n = k + l$. Note that $(\sigma, \sigma') \in \Sigma_k \times \Sigma_l$ sits canonically inside Σ_n with σ permuting the first k and σ' the last l elements of $\mathbf{n} = \{1, \dots, n\}$. Then $\mathscr{T}_*^\Sigma$ is a symmetric monoidal category with product

$$(X \wedge Y)_n = \bigvee_{k+l=n} \Sigma_{n+} \wedge_{\Sigma_k \times \Sigma_l} X_k \wedge Y_l,$$

where $(-)_+$ adds a disjoint basepoint. The twist map $X \wedge Y \cong Y \wedge X$ is described in Remark 2.1.5 [HSS00].

Remark. The category Σ is symmetric monoidal with respect to the sum

$$\mathbf{m} \sqcup \mathbf{n} = \{1, \dots, m, m+1, \dots, m+n\}$$

and the twist map $\mathbf{m} \sqcup \mathbf{n} \cong \mathbf{n} \sqcup \mathbf{m}$ sending $1 \leq k \leq m$ to $k + n$ and $m + 1 \leq k \leq m + n$ to $k - m$. The smash product $X \wedge Y$ as defined above is the Day convolution of X and Y with respect to $\sqcup$, as described in Appendix B.2. This gives some control over maps out of $X \wedge Y$, see Lemma B.2.1.

The sequence of spheres $S = (S^0, S^1, \dots)$ is a commutative monoid in this category, where Σ_n acts on S^n by permutation of the coordinates of a chosen decomposition

$$S^n \cong \underbrace{S^1 \wedge \cdots \wedge S^1}_{n}.$$

The commutative monoid S is called the *sphere spectrum*.

Definition 1.4.1. The category of *symmetric spectra* Sp^Σ is the category of (right) modules over the sphere spectrum S.

For an explicit description of the closed symmetric monoidal structure on the category of symmetric spectra see the standard texts mentioned above. With either of the stable model structures defined in the next section, the category of symmetric spectra is Quillen equivalent to Bousfield-Friedlander's category. This was shown in [HSS00, Section 4] for the category of simplicial symmetric spectra, which is Quillen equivalent to the category of topological simplicial spectra op. cit., Section 6.

Note. For a definition of the stable homotopy groups $\pi_*^S(X)$ for a symmetric spectrum, also written $\pi_*(X)$ or X_*, see Schwede [Sch08, Section 1].

Definition 1.4.2. If a symmetric spectrum R is a (commutative) algebra over S, we call R a (*commutative*) *ring spectrum*.

Remark. Note that if R is a (commutative) ring spectrum, then R_* is a graded (commutative) ring.

Remark. Let X be a symmetric spectrum with S-module structure $X \wedge S \to S$. By definition of $X \wedge S$ we see that this gives a map

$$(X \wedge S)_{n+1} = \bigvee_{k+l=n+1} \Sigma_{n+1,+} \wedge_{\Sigma_k \times \Sigma_l} X_k \wedge S^l \to X_{n+1}.$$

For $k = n$ and $l = 1$ we see that structure maps $\Sigma X_n = X_n \wedge S^1 \to X_{n+1}$ are part of the data of a symmetric spectrum.

The next example will be useful in constructing K_n from E_n.

Notation. Recall that a sequence $(x_0, \ldots, x_n)$ of elements in a ring R is called *regular* if multiplication for each $1 \leq k \leq n$ and all $k \leq i \leq n$, multiplication by x_i injective modulo the ideal $(x_1, \ldots, x_{k-1})$.

Example 1.4.3. Let R be a symmetric ring spectrum, and $I = (x_1, \ldots, x_n)$ a regular sequence in the coefficient ring $\pi_*^S R = R_*$. Then there is a new spectrum R/I with. See [Sch, Chapter 6].

Lemma 1.4.4 (Corollary 3.7, [Ang08])**.** *Let R be a symmetric ring spectrum and I a regular sequence of even degree generators in R_*. Then R/I is a ring spectrum.*

Note. For a symmetric spectrum model for the Eilenberg-Mac Lane spectrum HG representing singular cohomology with coefficients in G, see [Sch, Chapter 1].

Example 1.4.5. Let $X = X_0$ be a pointed space. We call X a *loop space* if it is homotopy equivalent to ΩY for some pointed space Y, and Y a *delooping* of X. If Y is itself a loop space, and so on, then X is an *infinite loop space*. An infinite loop space X_0 gives rise to a sequential spectrum picking homotopy equivalences

$$X_0 \simeq \Omega X_1 \simeq \Omega^2 X_2 \simeq \Omega^3 X_3 \simeq \dots$$

and letting $\Sigma X_n \to X_{n+1}$ be the adjoint of $X_n \simeq \Omega X_{n+1}$. Conversely, given a sequential spectrum X for which the adjoint of each structure map is a homotopy equivalence, X_0 is an infinite loop space.

Example 1.4.6. The space $BU \times \mathbf{Z}$ is the classifying space of complex vector bundles. Now, $\mathbf{Z} \times BU \simeq \Omega U$. Bott periodicity states that $U \simeq \Omega(\mathbf{Z} \times BU)$. This makes $BU \times \mathbf{Z}$ into an infinite loop space, and the associated sequential spectrum, KU, is the *periodic topological complex K-theory spectrum*. We construct a connective cover of KU as a symmetric ring spectrum in Chapter 2.

1.5 A positive stable model structure on symmetric spectra

The stable homotopy category of spectra was constructed by Boardman [Boa64] preceeding the construction by Bousfield-Friedlander [BF78] of a model category with a homotopy category equivalent to it. We want a stable model structure on the category of symmetric spectra which is Quillen equivalent, or related by a zig-zag of Quillen equivalences, to Bousfield-Friedlander's stable model category. There are many such model structures, notably the flat, injective and projective model structures on symmetric spectra. A construction of a stable model structure on simplicial symmetric spectra may be found in [HSS00]. While this induces a model structure for topological symmetric spectra, a more direct approach is used in [Hov01] to produce a stable model structure for symmetric spectra in any sufficiently structured category. This is the foundation used in the construction of a *positive* stable model structure on abstract symmetric spectra in [GG15], which is valid also for topological spectra. The main aspect of a positive model structure is that the homotopy properties of the level 0 of a spectrum map $f : X \to Y$ is largely ignored. The positive stable model structure constructed in this section is monoidal in the sense of [SS00], implying that it induces a model structure on monoids and modules, while positivity implies that it induces a model structure on the category *commutative* ring spectra. We first construct the non-positive stable model structure of [Hov01], and then the positive variant in [GG15].

The construction of both proceed in several steps. First by inducing a projective model structure on the category of topological symmetric spectra, whose weak equivalences are weak equivalences of spaces in each level. We then localize the projective model structure with respect to a class of maps which may be motivated as follows. By adjunction, $\mathscr{T}_*(\Sigma^n X, \Sigma^n Y) \cong \mathscr{T}_*(X, \Omega^n\Sigma^n Y)$, and so the stabilization sequence defining stable homotopy classes of maps above may be considered induced by a sequence of maps $\Omega^n\Sigma^n Y \to \Omega^{n+1}\Sigma^{n+1} Y$, and a stable class $X \to Y$ may be considered as a class of maps $X \to \Omega^n\Sigma^n Y$ for sufficiently large n. Construction of the positive stable model structure is done in much the same way, but ignoring level 0 of maps $X \to Y$, and the main theorem of [GG15] is this gives a model structure Quillen equivalent non-positive stable model category of [Hov01].

Definition 1.5.1 (Projective model structure)**.** Let $f : X \to Y$ be a map of symmetric spectra. Then f is a

- *level weak equivalence* if each $f_n : X_n \to Y_n$ is a weak equivalence in the Quillen model structure,
- a *level projective fibration* if $f_n : X_n \to X_n$ is a Serre fibration for each n, or
- a *projective cofibration* if it has the left lifting property with respect to every trivial level projective fibration,

i.e., a projective cofibration has the left lifting property level-wise with respect to Serre fribrations which are q-equivalences.

This defines a left proper and cellular model structure on Sp^Σ which is cofibrantly generated as follows [Hov01]. Consider the functor

$$\mathrm{ev}_n : Sp^\Sigma \to \mathscr{T}, \quad X \mapsto X_n.$$

It has a left adjoint F_n defined in two steps: Let $\widetilde{F}_n(X)$ be the symmetric sequence $\mathbf{n} \mapsto \Sigma_n \times X$ and $\mathbf{m} \mapsto *$ for $\mathbf{m} \neq \mathbf{n}$. Then $F_n(X) = \widetilde{F}_n(X) \wedge S$, the free S-module on $\widetilde{F}_n(X)$, where S is the symmetric sphere spectrum.

Let

- $I' = \bigcup_{n\geq 0} F_n I = \{F_n f : f \in I, n \geq 0\}$,
- $J' = \bigcup_{n\geq 0} F_n J = \{F_n f : f \in J, n \geq 0\}$, and
- $\mathscr{W}'$ = the category of level equivalences in Sp^Σ.

The projective model structure on Sp^Σ is $\mathscr{M} = (I', J', W')$. Let now $\zeta_n^X : F_{n+1}(X \wedge S^1) \to F_n(X)$ be the adjoint of the morphism $X \wedge S^1 \to \mathrm{ev}_{n+1} F_n X$, and let S be the set

$$S = \{\zeta_n^X | X \in \mathrm{dom}(I) \cup \mathrm{codom}(I), n \geq 0\}.$$

Define $\mathscr{M}_S$ to be the left Bousfield localization of $\mathscr{M}$ with respect to S, with generators $(I'_S, J'_S, \mathscr{W}'_S)$.

Definition 1.5.2. The model structure $\mathscr{M}_S$ is the *projective stable model structure* on Sp^Σ.

The *positive* projective stable model structure on Sp^Σ, which we now define, is Quillen equivalent to Sp^Σ with the model structure $\mathscr{M}_S$. Let

- $I'^+ = \bigcup_{n>0} F_n I = \{F_n f : f \in I, n > 0\}$,
- $J'^+ = \bigcup_{n>0} F_n J = \{F_n f : f \in J, n > 0\}$, and let
- $\mathscr{W}'^+ =$ the category whose morphisms are weak equivalences in level $n > 0$.

The difference between I', J', $\mathscr{W}'$ andI'^+, J'^+, W'^+ is that we ignore the 0-th level. Let $\mathscr{M}^+$ be the cofibrantly generated model structure generated by $(I'^+, J'^+, \mathscr{W}'^+)$. Write $S^+ = \{\zeta_n^X | X \in \mathrm{dom}(I) \cup \mathrm{codom}(I), n > 0\}$, where again the only difference between S^+ and the set S is that we look at ζ_n^X for $n > 0$ as opposed to $n \geq 0$. Let now $\mathscr{M}^+_{S^+}$ be the left Bousfield localization of $\mathscr{M}^+$ with respect to S^+, with generators $(I'^+_{S^+}, J'^+_{S^+}, \mathscr{W}'^+_{S^+})$.

Theorem 1.5.3 ([GG15]). *We have equalities of categories*

$$\mathscr{W}_S = \mathscr{W}^+_{S^+} = \mathscr{W}^+_S.$$

In other words the homotopy categories of the projective stable and positive projective stable model structures will be equivalent.

2 Higher K-theory

The algebraic K-theory of a variety was first developed by Grothendieck [Gro] and served as a key ingredient in a vast generalization of the Riemann-Roch theorem. Later ideas by Quillen [Qui73] allowed for the definition of higher K-groups by using ideas from topology. For a survey of Quillen's work on Higher algebraic K-theory see [Gra13]. Many constructions of higher K-theory now exist, notably the K-theory of a Waldhausen category. In this section we define a symmetric ring spectrum model for the K-theory of a commutative ring, and of a commutative ring spectrum. Some constructions of K-theory produce the K-groups $K_*(R)$ as the homotopy groups of a K-theory space, and some as the homotopy groups of a K-theory spectrum. A classical construction of the higher algebraic K-theory space of a commutative ring R is reasonably straightforward. The disjoint union $\coprod_{n\geq 0} BGL_n(R)$ is an associative monoid under block sum, and the K-theory space of R is

$$K(R)_{ésp} = \Omega B(\coprod_{n\geq 0} BGL_n(R)).$$

This is the free K-theory space of R, and roughly speaking it captures the stable properties of free R-modules and their automorphisms. Another construction due to Quillen [Qui69], homotopy equivalent to the previous one, is made from $BGL(R)$. The i-th homotopy group of this space is a candidate for the i-th K-group of R (for $i > 0$) with the essential issue that $\pi_1 BGL(R)$ is not usually abelian. The $+$-construction on this space is a new space, $BGL(R)^+$, whose fundamental group is the abelianization of $\pi_1 BGL(R)$ with all homology groups preserved. Then

$$K_i(R) = \pi_i(K_0(R) \times BGL(R)^+),$$

where $K_0(R)$ is the group associated to the commutative monoid of isomorphism classes of finitely generated projective R-modules. A similar construction via $\mathscr{I}$-spaces leads to a K-theory *space* $K(R)^{\mathscr{I}}$ associated to a ring spectrum [Sch04]. While a similar construction taking a symmetric ring spectrum to its K-theory *spectrum* $K(R)$ with underying space $K^{\mathscr{I}}(R)$ should exist somewhere in the literature, we develop instead the K-theory of a Waldhausen category. As a preparation for this very technical construction, we sketch the construction of a commutative ring spectrum model for the connective topological K-theory spectrum ku via Segal's Γ-spaces in the first and second section of this chapter.

2.1 Γ-spaces and associated symmetric spectra

Classical cohomology theories, i.e., singular cohomology with coefficients, are represented by deloopings of abelian groups. Γ-spaces were developed in [Seg74] and provide models for generalized cohomology theories. In particular, they are useful for associating a spectrum to a strict symmetric monoidal category or, a ring spectrum to some interpretation of a strict symmetric bimonoidal category (these are also called permutative and bipermutative categories, respectively). The category of Γ-spaces was later extended by Bousfield-Friedlander [BF78] in order to get a category with good homotopical properties, and the Γ-spaces of Segal are *special* Γ-spaces in this new category. In fact special Γ-spaces model E_∞-spaces, while group-like, or *very special*, Γ-spaces model infinite loop spaces, i.e., connective spectra.

Definition 2.1.1. Let Γ^{op} be the category of finite pointed sets, and denote by $\mathbf{n}_+$ the set $\{0, \ldots, n\}$ with basepoint 0. A Γ-*space* is a functor $A : \Gamma \to \mathscr{T}$ such that $A(\mathbf{0}_+)$ is contractible. A Γ-space A is called *special* if the map

$$p_n : A(\mathbf{n}_+) \to A(\mathbf{1}_+) \times \cdots \times A(\mathbf{1}_+),$$

induced by the n maps $\mathbf{n}_+ \to \mathbf{1}_+$ in Γ^{op} with support on a single element, is a weak equivalence.

Note. This definition of Γ is equivalent to that of Segal Γ^{op} in [Seg74], but more in line with later conventions.

Remark. We think of a Γ-space as a structure on the *underlying space* $A(\mathbf{1}_+)$.

Definition 2.1.2. We say that a Γ-space A is *group-like*, or *very special*, if the monoid of components $\pi_0 A(1_+)$ with multiplication defined by the composition

$$\pi_0 A(\mathbf{1}_+) \times \pi_0 A(\mathbf{1}_+) \xrightarrow{\pi_0(p_2^{-1})} \pi_0 A(\mathbf{2}_+) \xrightarrow{\pi_0(\mathbf{2}\to\mathbf{1})_+} \pi_0 A(\mathbf{1}_+),$$

where the second map is induced by the unique map $(\mathbf{2} \to \mathbf{1})_+$.

The symmetric spectrum associated to a Γ-space A Γ-space A extends to a functor on all pointed sets and pointed maps by forcing

$$A(U) = \operatorname{colim}_{V \subset U} A(V),$$

where V ranges over the finite subsets of U. This is a left Kan extension of A along the inclusion functor $\Gamma^{\mathrm{op}} \to \mathscr{E}_*$. Given a pointed simplicial set $X.$ we get a simplicial space by writing

$$[n] \mapsto A(X_n).$$

Denote by its geometric realization $A(X)$ (that is, we omit the index placeholder in $X.$). Note that A becomes a simplicial functor, and so we have natural maps

$$\mathrm{map}_*(X., Y.) \to \mathrm{map}_*(A(X.), A(Y.)).$$

Clearly Σ_n acts on $A(S^n)$. The symmetric sequence $(A(S^0), A(S^1), A(S^2), \dots)$ becomes symmetric sequence when given the structure maps $A(S^n) \wedge S^k \to A(S^{n+k})$ corresponding to $\mathrm{id} : S^n_. \wedge S^k_. \to S^n_. \wedge S^k_.$ under

$$\begin{aligned} s\mathscr{E}_*(S^n_. \wedge S^k_., S^n_. \wedge S^k_.) &\cong s\mathscr{E}_*(S^k_., \mathrm{map}_*(S^n_., S^n_. \wedge S^k_.)) \\ &\to s\mathscr{E}_*(S^k_., \mathrm{map}_*(A(S^n_.), A(S^n_. \wedge S^k_.))) \\ &\cong s\mathscr{E}_*(A(S^n_.) \wedge S^k_., A(S^n_. \wedge S^k_.)) \end{aligned}$$

as in [DGM13, Section 2.1.3]. The necessary $\Sigma_n \times \Sigma_k \subset \Sigma_{n+k}$-equivariance follows from naturality of the above correspondences and maps. We denote this spectrum by $\mathbf{B}A$, the *associated* spectrum.

2.2 Γ-categories and ku

Γ-spaces arise naturally from Γ-objects in small (topological) categories.

Definition 2.2.1. A Γ-*category* is a functor $\mathscr{C} : \Gamma^{\mathrm{op}} \to \mathrm{Cat}$ such that

- $\mathscr{C}(\mathbf{0}_+)$ is equivalent to the initial category $*$, and
- the map $p_n : A(\mathbf{n}_+) \to A(\mathbf{1}_+) \times \cdots \times A(\mathbf{1}_+)$, $\mathbf{n}_+ \to \mathbf{1}_+$ in Γ^{op} with support on a single element, is an equivalence of categories.

Remark. If $\mathscr{C}$ is a Γ-category, then $\mathbf{n}_+ \mapsto |N.\mathscr{C}(\mathbf{n}_+)|$ is a Γ-space denoted $|\mathscr{C}|$, and an associated symmetric spectrum $\mathbf{B}|\mathscr{C}|$ which we write $\mathbf{B}\mathscr{C}$.

Given a finite set S let $P(S)$ be the partially ordered set of subsets of S ordered by inclusion. A functor $P(S) \to \mathscr{C}$ is called an *S-cube* in $\mathscr{C}$. Let $\mathscr{C}$ be a category with finite sums $\oplus$ and initial object e. We make $\mathscr{C}$ into a Γ-category by letting $\mathscr{C}^{\oplus}(\mathbf{n}_+)$ be the category of $\mathbf{n}$-cubes in $\mathscr{C}$ sending $\emptyset$ to e, and taking disjoint unions to sums.

Remark. Given a pointed category $\mathscr{C}$ with finite sums, we get from this discussion an associated symmetric spectrum $\mathbf{B}\mathscr{C}^{\oplus}$.

Example 2.2.2. Let $\mathscr{V} = \mathscr{V}_{\mathbf{C}}$ be the category of finite dimensional vector spaces $\mathbf{C}$, and vector space isomorphisms. Then $\mathscr{C}$ has a sum $\oplus$. This category is equivalent the category $\mathscr{V}' = \mathscr{V}'_{\mathbf{C}}$ with objects $\mathbf{C}^k$, $k \geq 0$, and the associated spectrum will be equivalent. We can define $ku := \mathbf{B}\mathscr{V}^{\oplus}$ or $ku := \mathbf{B}\mathscr{V}'^{\oplus}$.

Remark. In order to make ku into a ring symmetric ring spectrum we need a unit $S \to ku$, and and a map $ku \wedge ku \to ku$ giving ku the structure of an S-algebra. In order to get a map out of a smash product, we know from Lemma B.2.1 that we need appropriatly $\Sigma_k \times \Sigma_l \subset \Sigma_{k+l}$-equivariant maps

$$ku_k \wedge ku_l \to ku_{k+l}.$$

This is possible using the tensor product $\otimes_{\mathbf{C}}$ for $\mathscr{V}_{\mathbf{C}}$. Segal sketches a construction as follows.

Definition 2.2.3. A *multiplication* on a group-like Γ-space A is a functor

$$\hat{A} : \Gamma \times \Gamma \to \mathscr{T}$$

together with natural transformations $i_1 : \hat{A}(\mathbf{m}, \mathbf{n}) \to A(\mathbf{m})$, $i_2 : \hat{A}(\mathbf{m}, \mathbf{n}) \to A(\mathbf{n})$, and $m : \hat{A}(\mathbf{m}, \mathbf{n}) \to A(\mathbf{m} \times \mathbf{n})$, such that $(i_1, i_2) : \hat{A}(\mathbf{m}, \mathbf{n}) \to A(\mathbf{m}) \times A(\mathbf{n})$ is a weak equivalence for every $\mathbf{m}, \mathbf{n}$.

We produce a multiplication on $\mathscr{V}^{\oplus}$ using $\otimes = \otimes_{\mathbf{C}}$ as follows. Let S and T be finite sets. Denote by $\mathscr{V}^{\oplus,\otimes}(S, T)$ the category with objects tuples (X, Y, Z, μ), where

- X is an S-cube in $\mathscr{V}$,
- Y is a T-cube in $\mathscr{V}$, and
- Z is a $S \times T$-cube in $\mathscr{V}$,
- X, Y, Z all take disjoint unions to direct sums,

- μ is a collection of bilinear maps $\mu_{U,V} : X(U) \times Y(V) \to Z(U \times V)$ for all pairs of subset $U \subset S$ and $V \subset T$, expressing $Z(U \times V)$ as the tensor product $X(U) \otimes Y(V)$.[1]

Then

$$(\mathbf{m}_+, \mathbf{n}_+) \mapsto |N.\mathscr{V}^{\oplus,\otimes}(\mathbf{m}, \mathbf{n})|$$

is a multiplication on the Γ-space $\mathscr{V}^{\oplus}$. The statement is roughly that, though modern clarifications are necessary, this multiplication induces a (symmetric) ring spectrum map $ku \wedge ku \to ku$.

Remark. For associativity and commutativity we need higher multiplications

$$A_k : \underbrace{\Gamma \times \cdots \times \Gamma}_{k} \to \mathscr{T},$$

where $A_1 = A, A_2 = \hat{A}$, etc., together with natural transformations encoding these relations.

2.3 The symmetric $S.$-construction

A Waldhausen category is essentially a category with a category of weak equivalences $w\mathscr{C}$, a class of cofibrations $A \to B$ (sub-objects) and existence of quotient objects B/A. In this setup we can define K_0 of $\mathscr{C}$ to be free additive group on weak equivalence classes modulo the relation $[A] + [B/A] = [B]$. From there, $K_0(R) := K_0(\mathscr{C})$ where $\mathscr{C}$ is a Waldhausen category of finitely generated projective modules over R. The higher K-theory of $\mathscr{C}$ should follow the same principle. Given a Waldhausen category $\mathscr{C}$ we can define a simplicial category $S.\mathscr{C}$ whose n-simplices are n-fold filtrations

$$A_0 \to A_1 \cdots \to A_n = A$$

of objects of $A \in \mathscr{C}$ together with choices of filtration quotients. This is itself Waldhausen category and the construction may be iterated, $S.^{(n)}\mathscr{C}$, and the n-th K-theory space is $K(R)_n = |\operatorname{diag} N.wS.^{(n)}\mathscr{C}|$. There are also structure maps $|\operatorname{diag} N.wS.^{(n)}\mathscr{C}| \wedge S^1 \to |\operatorname{diag} N.wS.^{(n+1)}\mathscr{C}|$. In order for Σ_n to act on $K(R)_n$, and the structure maps $\Sigma^k K(R)_n \to K(R)_{n+k}$ to be equivariant, some care must be taken.

The original definition of the K-theory of a Waldhausen category is due to Waldhausen [Wal85] before the invention of symmetric spectra, but he does indeed construct a

[1] In other words, $\mu_{U,V}$ is initial with respect to bilinear maps out of $X(U) \times Y(V)$.

symmetric positive Ω-spectrum. For a very accessible account of this, see [Boy], which is compiled from [GH99]. For convenience we record here the main constructions to produce $K(R)$. Unpublished notes by Rognes [Rog] and Rognes' thesis [Rog92] have also proved helpful. In [Dun14] it is shown that $K(R)$ is a commutative ring spectrum.

Definition 2.3.1. A *Waldhausen category* is a category $\mathscr{C}$ with zero object 0 together with a subcategory $c\mathscr{C}$ of morphisms called *cofibrations*, and a subcategory $w\mathscr{C}$ whose morphisms are called *weak equivalences*, such that

1. isomorphisms are cofibrations,

2. the unique morphism from the initial object is a cofibration (every object is 'cofibrant'),

3. the pushout of a cofibration along any morphism exists and is a cofibration, and

4. for any commutative diagram

$$\begin{tikzcd} D \arrow[d, "\sim"'] & A \arrow[l] \arrow[r, tail] \arrow[d, "\sim"] & B \arrow[d, "\sim"] \\ D' & A' \arrow[l] \arrow[r, tail] & B' \end{tikzcd}$$

   where the vertical maps are weak equivalences and the right-most horizontal maps are cofibrations, then the induced map $B \cup_A D \to B' \cup_{A'} D'$ is a weak equivalence.

For a cofibration $A \rightarrowtail B$ denote by B/A a choice of quotient $B \cup_A 0$.

Definition 2.3.2. A functor $\mathscr{C}' \to \mathscr{C}$ between Waldhausen categories is called *exact* if it preserves the zero object, cofibrations, and pushouts along cofibrations. A functor $F : \mathscr{C}' \times \mathscr{C}'' \to \mathscr{C}$ is called *biexact* if it is exact in each variable, i.e., $F(A, -)$ and $F(-, B)$ are exact functors for all $A \in \mathscr{C}'$ and $B \in \mathscr{C}''$.

Definition 2.3.3. Let Q be a finite set, and $P(Q)$ be the power-set of Q ordered by inclusion, considered as a category. A *Q-cube* in $\mathscr{C}$ is a functor $X : P(Q) \to \mathscr{C}$, and it is a *cofibration cube* if for each $S \subset T \subset Q$ the canonical map

$$\operatorname{colim}_{S \subseteq U \subsetneq T} X(U) \to X(T)$$

is a cofibration in $\mathscr{C}$. This is also called a *lattice* in [Rog92].

Example 2.3.4. The category Γ^{op} of finite pointed sets and pointed maps is a Waldhausen category with injections the cofibrations, and bijections the weak equivalences. Further, the smash product

$$\wedge : \Gamma \times \Gamma \to \Gamma, \quad (\mathbf{m}_+, \mathbf{n}_+) \mapsto \mathbf{m}_+ \wedge \mathbf{n}_+ = (\mathbf{mn})_+$$

is biexact.

Definition 2.3.5. If Q is a finite set, denote by Δ^Q a $|Q|$-fold product of the simplex category Δ. Objects of Δ^Q are tuples of positive integers $[n_Q] = [n_s], s \in Q$, and morphisms are also products $[n_s] \to [m_s]), s \in Q$. We call Δ^Q the *Q-simplex category*. A *Q-simplicial object* in a category $\mathscr{C}$ is a functor $(\Delta^Q)^{\mathrm{op}} \to \mathscr{C}$.

Definition 2.3.6. Consider the partially ordered set $[n] = \{0, \ldots, n\}$ as a category. For for $[n_Q]$ in Δ^Q define a Q-cube in the category of arrows $\mathrm{Ar}[n_Q]$ of $[n_Q]$. For a morphism $i \to j = (i_s \to j_s), s \in Q$ in $[n_Q]$ and a subset $U \subset Q$ let $(i \to j)_U$ have components $i_s \to j_s$ for $s \in U$ and identities $i_s \to i_s$ for $s \notin U$. Then $U \mapsto (i \to j)_U$ is a Q-cube in $\mathrm{Ar}[n_Q]$.

For a Q-indexed tuple $n_Q = (n_s), s \in Q$, denote by $S^Q_{n_Q}\mathscr{C}$ the following full subcategory of functors $\mathrm{Ar}[n_Q] \to \mathscr{C}$ consisting of functors $(i \to j) \mapsto A_{i \to j}$ such that:

1. if some component $i_s \to j_s$ of $i \to j$ is the identity, then $A_{i \to j} = 0$,

2. for every pair of composable morphisms $i \to j \to k$, the Q-cube

$$U \mapsto A_{(j \to k)_U \circ (i \to j)}$$

is a cofibration cube and,

3. the square

$$\begin{tikzcd}
\operatorname{colim}_{U \subsetneq Q} A_{(j \to k)_U \circ (i \to j)} \arrow[r, rightarrowtail] \arrow[d] & A_{i \to k} \arrow[d] \\
0 \arrow[r] & A_{j \to k}
\end{tikzcd}$$

is cocartesian, that is to say, $A_{j \to k} \cong A_{i \to k} / \operatorname{colim}_{U \subsetneq Q} A_{(j \to k)_U \circ (i \to j)}$.

Remark. The objects of $S^Q_{n_Q}\mathscr{C}$ among functors $A : \mathrm{Ar}[n_Q] \to \mathscr{C}$ are the restrictions $\sigma^*(A)$ along the functor $[n_Q] \to \mathrm{Ar}[n_Q]$ sending j to $0 \to j$.

Remark. For closer examination of the data of such functors see [Boy].

We make $S^Q_{n_Q}\mathscr{C}$ a Q-simplicial object in Waldhausen categories as follows. A map $f : A \to A'$ in $S^Q_{n_Q}\mathscr{C}$ is a cofibration if for every pair of composable morphisms $i \to j \to k$ in $[n_Q]$ the induced map of Q-cubes

$$\big(U \mapsto A_{(j\to k)_U \circ (i \to j)}\big) \to \big(U \mapsto A'_{(j\to k)_U \circ (i \to j)}\big)$$

is a cofibration cube when viewed as a $(|Q|+1)$-cube. A morphism $f : A \to A'$ is a *weak equivalence* if it is a weak equivalence in $\mathscr{C}$ on each component. If $Q = \mathbf{1} = \{1\}$ write $S^Q_{\cdot}C = S_{\cdot}C$ (which is equivalent to Waldhausen's original construction).

The symmetric spectrum $K(\mathscr{C})$. By functoriality in the automorphisms of $\mathbf{n}$, Σ_n acts on $S^{\mathbf{n}}_{\cdot}\mathscr{C}$. Let $K(\mathscr{C})$ be the based symmetric sequence

$$K(\mathscr{C}) : \mathbf{n} \mapsto K(\mathscr{C})_n = |\operatorname{diag} N_{\cdot}wS^{\mathbf{n}}_{\cdot}\mathscr{C}|$$

with basepoint the constant functor $i \to j \mapsto *$. We define $\Sigma_n \times \Sigma_m \subset \Sigma_{n+m}$-equivariant structure maps $K(\mathscr{C})_n \wedge S^m \to K(\mathscr{C})_{n+m}$ making $K(\mathscr{C})$ a symmetric spectrum. If $P \subset Q$ there is an isomorphism of Q-simplicial Waldhausen categories

$$S^Q_{\cdot} \cong S^{Q\setminus P}_{\cdot}(S^P_{\cdot}\mathscr{C}),$$

so ordering Q we get

$$S^Q_{\cdot}C \cong \underbrace{S_{\cdot} \cdots S_{\cdot}}_{|Q| \text{ times}}\mathscr{C}.$$

This isomorphism is appropriately equivariant. Therefore it is enough to define structure maps in the case $Q = \mathbf{1}$, i.e., for $N_{\cdot}S_{\cdot}\mathbf{C}$. Then

$$\operatorname{ev}^V_1 N_{\cdot}wS_{\cdot}\mathscr{C} = N_{\cdot}wS_1\mathscr{C} \cong N_{\cdot}w\mathscr{C}$$

and $\operatorname{ev}^v_0 N_{\cdot}wS_{\cdot}\mathscr{C} = *$, where ev^v_n is evaluation in the 'vertical' simplicial direction $S_{\cdot}$. By the adjuction $- \times \Delta^n \dashv \operatorname{ev}^V_n$ we get a map

$$N_{\cdot}w\mathscr{C} \times \Delta^1 \to N_{\cdot}wS_{\cdot}\mathscr{C}$$

which factors through $N_{\cdot}w\mathscr{C} \times \Delta^1 \to N_{\cdot}w\mathscr{C} \wedge \Delta^1/\partial\Delta^1$.

Pairings of K-theory. Let $\mathscr{C}', \mathscr{C}''$ and $\mathscr{C}$ be Waldhausen categories and let

$$\wedge : \mathscr{C}' \times \mathscr{C}'' \to \mathscr{C}$$

be a biexact functor. If Q, Q' are disjoint finite sets, then this induces a biexact functor

$$\wedge : S_{\cdot}^{Q}\mathscr{C}' \times S_{\cdot}^{Q'}\mathscr{C}'' \to S_{\cdot}^{Q\cup Q'}\mathscr{C}$$

by setting

$$(A \wedge A')_{i\cup i' \to j \cup j'} = A_{i\to j} \wedge A'_{i'\to j'}.$$

Picking $Q = \mathbf{m} = \{1, \dots m\}$ and $Q' = \{n+1, \dots, n+m\}$ we get, after passing to the classifying space of the subcategory of weak equivalences, a $\Sigma_m \times \Sigma_n$-equivariant map

$$K(\mathscr{C}')_m \times K(\mathscr{C}'')_n \to K(\mathscr{C})_{m+n}$$

which factors through $K(\mathscr{C}')_m \wedge K(\mathscr{C}'')_n$. When $\mathscr{C} = \mathscr{C}' = \mathscr{C}''$ this endows $K(\mathscr{C})$ with the structure of a ring spectrum. If $\wedge$ is commutative and associative, then so is $K(\mathscr{C})$.

Example 2.3.7. The symmetric spectrum $K(\Gamma)$ is stably equivalent to the sphere spectrum S, and $\wedge : \Gamma \times \Gamma \to \Gamma$ realizes the commutative monoid $K(\Gamma)$ as the sphere spectrum. This is the Barratt-Priddy-Quillen theorem [BP72].

Example 2.3.8 (The symmetric ring spectrum $K(R)$)**.** Let R be a commutative symmetric ring spectrum. Then the category $\mathscr{M}_R$ of modules over R is a symmetric monoidal model category. Then the category $c\mathscr{M}_R^{\text{fin}}$ of finite[2] cofibrant objects in $\mathscr{M}_R$ is a Waldhausen category, and the *algebraic K-theory spectrum of R* is given by $K(R)_n = |N.wS_{\cdot}^{\mathbf{n}}c\mathscr{M}_R|$. The smash product $\wedge_R$ gives $K(R)$ the structure of a commutative ring spectrum.

Example 2.3.9. In order to get the free algebraic K-theory of R, we may restrict ourself to the following category. Let $\mathscr{M}_R^{\text{free}}$ be the category whose objects are the integers $1, 2, \dots,$ and whose morphisms are

$$\mathscr{M}_R^{\text{free}}(m, n) = \mathscr{M}_R(R^{\vee m}, R^{\vee n}).$$

This category inherits the structure of a Waldhausen category from the model structure on $\mathscr{M}_R$.

2.4 The inclusion $BGL_1(ku) \to K^{\mathscr{I}}(ku)$

In this section we follow Schlichtkrull [Sch04] in defining the K-theory space $K^{\mathscr{I}}(R)$ of a (commutative) symmetric ring spectrum, and defining an inclusion

$$BGL_1(R) \to K(R).$$

[2] In the sense of Appendix A.2.

For $R = ku$, $B^3\mathbf{Z} = K(\mathbf{Z}, 3)$ sits inside $BGL_1(ku)$ and justifies our interest in the third Eilenberg-Mac Lane space of the integers.

Write $\mathscr{I}$ for the category of finite sets $\mathbf{n} = \{1, \ldots, n\}$ and injections, i.e., a skeleton of the category of finite sets and injections. It is a symmetric monoidal category under disjoint union $\sqcup$, with twist the shuffle $\mathbf{m} \sqcup \mathbf{n} \to \mathbf{n} \sqcup \mathbf{m}$.

Definition 2.4.1. An *$\mathscr{I}$-space* is a functor $\mathscr{I} \to \mathscr{T}$. The category of $\mathscr{I}$-spaces is a symmetric monoidal category with respect to Day convolution in diagram categories. An *$\mathscr{I}$-space monoid* is a monoid in the category of $\mathscr{I}$-spaces with respect to this monoidal structure.

The $\mathscr{I}$-space monoid $\Omega^{\mathscr{I}}(R)$. Given a symmetric ring spectrum R we define an $\mathscr{I}$-space monoid $\Omega^{\mathscr{I}}$ as follows. Write $\overline{\alpha}$ for the map $n = \mathbf{l} \sqcup \mathbf{m} \to \mathbf{n}$ that is order-preserving on $\mathbf{l}$ and α on $\mathbf{m}$. Let $\Omega^{\mathscr{I}}$ be the $\mathscr{I}$-space $\mathbf{n} \mapsto \Omega^n R_n$, where $\alpha : \mathbf{m} \to \mathbf{n} \mapsto \Omega^m R_m \to \Omega^n R_n$ define by mapping $f \in \Omega^m R_m$ to the composition

$$S^n \xrightarrow{\overline{\alpha}^{-1}} S^n \cong S^l \wedge S^m \xrightarrow{1 \wedge f} S^l \wedge R_m \xrightarrow{\sigma^l} R_n \xrightarrow{\overline{\alpha}} R_n,$$

where the action on S^n is the usual left action given by permutation of the coordinates of $S^n \cong (S^1)^{\wedge n}$. A monoidal structure

$$\mu_{m,n} : \Omega^m R_m \times \Omega^n R_n \to \Omega^{m+n} R_{m+n}$$

is defined as the composition

$$\mu_{m,n}(f, g) :\; S^m \wedge S^n \xrightarrow{f \wedge g} R_m \wedge R_n \xrightarrow{\mu_{m+n}} R_{m+n}$$

Let $\Omega^n R_n^*$ be the union of components of $\Omega^n R_n$ with stable multiplicative inverses in the sense that for each $f \in \Omega^n R_n^*$ there is an $g \in \Omega^m R_m$ with $\mu(f, g) \simeq 1_{m+n} \in \Omega^{m+n} R_{m+n}$.

Definition 2.4.2. The *units* of R is the associative monoid $GL_1(R) = \operatorname{hocolim}_{\mathscr{I}} \Omega^n R_n^*$.

Remark. If R is commutative $GL_1(R)$ is an infinite loop space.

Let $M_n(R)_m = \operatorname{map}(\mathbf{n}_+, \mathbf{n}_+ \wedge R_m) \cong \prod_{\mathbf{n}} \bigvee_{\mathbf{n}} R_m$ be a ring spectrum with multiplication resembling matrix multiplication, see e.g., Schwede [Sch, Chapter 1, Section 3, Example 3.44]. Define $GL_n(R) = GL_1(M_n(R))$ with monoid structure from the multiplication in $M_n(R)$. There is an additional direct sum structure

$$\operatorname{Map}_*(\mathbf{m}_+ \wedge S^k, \mathbf{m}_+ \wedge R_k) \times \operatorname{Map}_*(\mathbf{n}_+ \wedge S^l, \mathbf{n}_+ \wedge) \to \operatorname{Map}_*((\mathbf{m} \sqcup \mathbf{n})_+ \wedge R_{k+l}).$$

This gives $\coprod_{n\geq 0} BGL_n(R)$ the structure of an associative topological monoid, and the K-theory space of R is its group completion, i.e.,

$$K^{\mathscr{I}}(R) := \Omega B(\coprod_{n\geq 0} BGL_n(R)).$$

This is the free K-theory of R. Embed $BGL_1(R)$ in the 1-simplexes of the simplicial set $B.(\coprod_{n\geq 0} BGL_n(R))$. The realization of its adjoint is a map

$$S^1 \wedge BGL_1(R)_+ \to B(\coprod_{n\geq 0} BGL_n(R)).$$

Adjunction gives an unbased map $BGL_1(R) \to K^{\mathscr{I}}(R)$.

Remark. An approach involving the $S.$ construction should also be possible. The Waldhausen category of finite sums of ku and self equivalences should give free K-theory of ku, and realization of the nerve of self equivalences of ku gives $BGL_1(ku)$. We did not have opportunity to make this precise.

3 Complex oriented spectra

Complex oriented spectra may be considered spectra with a good theory of Chern classes. Their most notable feature is perhaps that they have an associated formal group law, which we define below and is essentially a one-dimensional commutative formal affine group scheme with coordinate. It governs the behaviour of the Chern class of a tensor product of line bundles in terms of the Chern classes of its tensorands. There are many examples, including $H\mathbf{Z}$, $H\mathbf{F}_p$, KU, MU (defined below), and K_n (also defined below). Quillen shows that any formal group law gives rise to an MU_*-module [Qui69], which in turn gives rise to a functor on spaces. The Landweber exact functor theorem tells us when this functor is a homology theory, namely when the MU_*-module is *Landweber exact*. Over a separably closed field of characteristic $p > 0$ a formal group law is classified up to isomorphism by its height. The *Honda formal group law* in height n is a particularly simple representative. While the Honda formal group law is not Landweber exact, its universal deformation is. Work by Hopkins-Miller shows that the associated homology theory lifts to a structured E_∞ ring spectrum E_n, for which there is a commutative symmetric ring spectrum model. In this section we develop the main definitions of complex oriented spectra and give a model for MU. We state a variant of the Landweber exact functor theorem, and develop the theory of formal group laws far enough to state the existence of the Lubin-Tate spectra E_n. Then we construct K_n from E_n. Proofs are omitted, but the material on complex oriented theories and formal groups may be found in Ravenel [Rav86] or Adams [Ada95], as well as Rezk [Rez98]. A proof of the Hopkins-Miller theorem, as well as proofs of statements on deformations of formal group laws, may be found in [Rez98].

3.1 Complex orientations and formal group laws

Definition 3.1.1. Let E be a ring spectrum with cohomology theory $E^*(-)$. A *complex orientation* of E is a class $x \in \widetilde{E}^2(\mathbf{C}P^\infty)$ that restricts to a unit in $\widetilde{E}^2(S^2)$ (other conventions are possible). The spectrum E together with a choice of complex orientation x is called a *complex oriented* spectrum, written (E, x).

Proposition 3.1.2. *If (E, x) is complex oriented, then $E^*(\mathbf{C}P^\infty)$ is a graded power series ring $E^*[\![x]\!]$. Further, we have an isomorphism*

$$E^*(\mathbf{C}P^\infty \times \mathbf{C}P^\infty) \cong E^*[\![x \otimes 1, 1 \otimes x]\!].$$

By abuse of notation we usually write $x = x \otimes 1$ and $y = 1 \otimes x$, so that $E^*[\![x \otimes 1, 1 \otimes x]\!]$ is written $E^*[\![x, y]\!]$.

Definition 3.1.3. Let A be a graded ring. A (1-*dimensional, commutative*) *formal group law* over A is a formal power series $F \in A[\![x, y]\!]$ such that

- $F(x, y) = x + y +$ higher order terms,
- $F(x, F(y, z)) = F(F(x, y), z)$, and
- $F(x, y) = F(y, x)$.

A formal group law has an associated univariate power series ι such that $F(x, \iota(x)) = 0$. We may inductively define $[n](x) = F(x, [n-1](x))$ with $[1](x) = x$.

Example 3.1.4. The *additive* group law F_a is given by $F_a(x, y) = x+y$. The *multiplicative* formal group law is $F_m(x, y) = x + y + axy$.

If E is complex oriented, the multiplication $\mu : \mathbf{C}P^\infty \times \mathbf{C}P^\infty \to \mathbf{C}P^\infty$ representing the tensor product gives a bivariate formal power series by setting $F = \Delta^*(x) \in E^*[\![x, y]\!]$. The formal power series F is a formal group law over E^*. This follows from the properties of μ.

Example 3.1.5. Both $H\mathbf{Z}$ and KU are complex oriented theories. Their associated formal group laws are F_a and F_m respectively.

3.2 MU and the Landweber exact functor theorem

The following construction of MU is from Schwede's unpublished book project on symmetric spectra [Sch, Chapter 1].

Remark. Let V be a finite-dimensional real vector space. We mean by S^V the one-point compactification of V. If V is of dimension n, S^V is an n-dimensional sphere, and given an ordered basis permuation of the coordinates of V determines a based Σ_n action on $S^V \cong S^n$. If $V = V' \oplus V''$, then $S^V \cong S^{V'} \wedge S^{V''}$

Definition 3.2.1. Let $EU(n)$ be the universal principal bundle over $BU(n)$. Let

$$\overline{MU}_n = EU(n)_+ \wedge_{BU(n)} S^{\mathbf{C}^n}$$

be the Thom construction on the universal vector bundle $EU(n) \times_{BU(n)} \mathbf{C}^n$ over $BU(n)$. It is a Σ_n space by conjugation, permuting the coordinates on $\mathbf{C}^n$, and the $\Sigma_p \times \Sigma_q \subset \Sigma_{p+q}$ equivariant identification $\mathbf{C}^p \oplus \mathbf{C}^q \to \mathbf{C}^{p+q}$ induces a multiplication

$$\mu_{p,q} : \overline{MU}_p \wedge \overline{MU}_q \to \overline{MU}_{p+q}.$$

Set now $MU_n = \Omega^n \overline{MU}_n$ and define

$$MU_p \wedge MU_q \to MU_{p+q}$$

by $f \wedge g \mapsto \mu_{p,q} \circ (f \wedge g)$, which determines a pairing $MU \wedge MU \to MU$ by Lemma B.2.1. The S-module structure on MU is given by the identity $S^0 \to S^0 \cong MU_0$, and the map $S^1 \to MU_1$ which is given by regarding $\mathbf{C}$ as $\mathbf{R} \cdot 1 \oplus \mathbf{R} \cdot 1$, such that $S^{\mathbf{C}} \cong S^{\mathbf{R}\cdot 1} \wedge S^{\mathbf{R}\cdot i}$ and taking the adjoint of $S^{\mathbf{R}\cdot 1} \wedge S^{\mathbf{R}\cdot i} \cong S^{\mathbf{C}} \to \overline{MU}_2$. The commutative symmetric ring spectrum MU is the *complex cobordism* spectrum.

Milnor and Novikov [Mil60; Nov60; Nov62] showed that the coefficients of MU, which classifies cobordism classes of stably (almost) complex manifolds, is isomorphic to

$$\mathbf{Z}[x_1, x_2, \dots], \quad |x_i| = 2i.$$

There is a ring L, called the *Lazard ring*, together with a universal formal group law $F_U \in L[\![x, y]\!]$, for which L corepresents formal group laws in the sense that if F is a formal group law over a (commutative) ring A, there is a unique homomorphism $f : L \to A$ such that $F = f_*(F_U)$, where $f_*(F_U)$ is defined by applying f to the coefficients of F.

Remark. What we call f_* here is called f^* elsewhere in the literature. This is because it is convenient to work in the 'opposite' geometric language of affine formal groups. In particular when discussing the relation between the classifying stack of elliptic curves and the classifying stack of formal group laws.

Quillen showed that MU_* is canonically isomorphic to L, and that the formal group law associated to MU, F_{MU}, is the universal formal group law under this identification [Qui69]. As such, a formal group law over a ring A, provides A with an MU_*-module structure via the ring homomorphism $MU_* \cong L \to A$. Given an MU_*-module M one may form the functor

$$X \mapsto MU_*(X) \otimes_{MU_*} M$$

on finite CW complexes. The Landweber exact functor theorem [Lan76] tells us when this is a homology theory.

Theorem 3.2.2 (Landweber, Corollary 2.7)**.** *Let M be an MU_*-module such that for each prime p the sequence $(p, x_{p-1}, \dots, x_{p^n-1})$ is regular for M. Then $MU_*(X) \otimes_{MU_*} M$ is a homology theory on CW complexes.*

3.3 The Honda formal group law and its universal deformation

In this section we first develop some theory on formal group laws after [Rav86, Appendix A2]. Then, following [Rez98], we provide necessary background for the Hopkins-Miller theorem, which is also proved in [Rez98]. Let F and G be formal group laws over a ring A. Then a homomorphism $F \to G$ is a power series $f \in A[\![x]\!]$ with vanishing constant term such that

$$f(F(x, y)) = G(f(x), f(y)).$$

The isomorphisms this with non-zero leading term ax with a a unit. Such an isomorphism is a *strict* isomorphism if the linear term is 1. A strict isomorphism of F with the additive formal group is called a *logarithm* for F. A formal group over a torsion-free $\mathbf{Z}_{(p)}$-algebra is p-*typical* if has a logarithm of the form

$$\log(x) = \sum_{i \geq 0} \ell_i x^{p^i}.$$

If F is a formal group law and the leading non-zero term of its$[p]$-series $[p](x)$ is ax^{p^n}, the integer n is an isomorphism invariant called the *height* of F. There is a unique p-typical formal group law $F = F_n$ of height n over $\mathbf{Z}_{(p)}$ with p-series $[p](x) = x^{p^n}$. We call its reduction mod p, written H_n, the *Honda* formal group law after Honda [Hon70].

The groupoid of deformations. We turn to deformations of formal group laws. Let (K, F) be a pair with K a field of characteristic $p > 0$ and F a formal group law over K. A *deformation* of (K, F) to a local ring $(A, \mathfrak{m})$ with canonical projection $\pi : A \to A/\mathfrak{m}$, is a pair (G, i) with G a formal group over A and i a homomorphism $K \to A/\mathfrak{m}$ such that $i_*(F) = \pi_*(G)$.

Let (G_1, i_1) and (G_2, i_2) be two deformations of (K, F) to $(A, \mathbf{m})$. When $i_1 = i_2 = i$ we define a morphism $f : (G_1, i) \to (G_2, i)$ of deformations to be an isomorphism $f : G_1 \to G_2$ of formal group laws over A such that $\pi_*(f)$ is the identity of

$$i(F) = \pi_*(G_1) = \pi_*(G_2).$$

Denote the resulting groupoid by $\mathrm{Def}_K(A)$. It is functorial in A.

Definition 3.3.1. Let F be a formal group over K and consider the functor

$$A \mapsto \pi_0 \operatorname{Def}_F(A),$$

taking a complete local ring $(A, \mathfrak{m})$ to the set of isomorphism classes of deformations of F to A. Then a formal group $(\widehat{F}, \widehat{i})$ over a complete local ring B is a *universal deformation* of F if the functor $\pi_0 \operatorname{Def}_F(A)$ is corepresented by B, i.e., to any isomorphism class $[(G, i)]$ there is a unique K-algebra map $B \to A$ for which $[(G, i)] = [(f_*(\widehat{F}), i)]$.

Theorem 3.3.2 (Part of Theorem 4.4 [Rez98], due to Lubin-Tate)**.** *Let F be a formal group law of finite height n over K, Then there is a complete local ring $(E(K, F), \mathfrak{m})$ with an isomorphism $i : K \to E(K, F)/\mathfrak{m}$ and a formal group law $\widehat{F}$, such that the pair $(\widehat{F}, i)$ is a universal deformation of F.*

3.4 E_n and K_n

Denote by $\mathbf{W}K$ the ring of Witt vectors on K. Let $K = \mathbf{F}_{p^n}$ and H be the Honda formal group law pushed forward from $\mathbf{F}_p$. Then for $K = \mathbf{F}_{p^n}$

$$E(K, H) = \mathbf{W}\mathbf{F}_{p^n}[\![u_1, \ldots, u_{n-1}]\!].$$

The homology theory associated to universal deformation $\widehat{H}$ of H, over

$$\mathbf{W}\mathbf{F}_{p^n}[\![u_1, \ldots, u_{n-1}]\!],$$

is Landweber exact. Further, the associated homology theory lifts to an E_∞ ring spectrum, in the sense of [Lew+86]. Such a spectrum has an associated symmetric spectrum with E_∞ action, which may be rigidified to a symmetric ring spectrum. This is recorded in the following theorem.

Theorem 3.4.1 (Hopkins-Miller)**.** *There is a commutative symmetric ring spectrum E_n for which:*

- *the coefficients are concentrated in even degrees, and*
- *there is a unit $u \in E_n^2$, which may be taken as a complex orientation of E_n. Finally,*
- *the corresponding formal group law over $\pi_0 E_n$ is the universal deformation of H.*

There is an associated symmetric spectrum, which we denote by E_n and call the *Lubin-Tate spectrum* in *height* n.

We define $K_n = E_n/(p, u_1, \ldots, u_{n-1})$ as described in [Sch, Chapter 1]. Lemma 1.4.4 applies to show that K_n is a symmetric ring spectrum.

4 Algebraic preliminaries

4.1 Hopf algebras and (co)homology

The study of Hopf algebras was initiated by Hopf [Hop41] in order to study the cohomology of Lie groups with field coefficients. One reason for their usefulness in computations of generalized cohomology, as in the work of Ravenel-Wilson [RW80], is that differentials in a spectral sequence are severely restricted by the presence of a compatible Hopf algebra structure. For the main definitions and results see Milnor-Moore [MM65] or Sweedler [Swe69]. Hopf algebras are not to be confused with the unfortunately named Hopf rings [RW76], which are roughly graded rings in the category of graded coalgebras.

Definition 4.1.1 (Hopf algebra)**.** Let K be a graded commutative ring, and write $\otimes = \otimes_K$. Let H be a unital graded commutative K-algebra, with multiplication μ and unit η, together with a counital coassociative cocommutative coalgebra structure ($\psi : H \to H \otimes H$ and $\epsilon : H \to K$), and a map $\chi : H \to H$ called *conjugation*. If these structures are subject to the relation

$$\psi\mu = (\mu \otimes \mu)(1 \otimes \tau \otimes 1)(\psi \otimes \psi),$$

stating ψ is an algebra homomorphism, and

$$\eta\epsilon = \mu(\chi \otimes 1)\psi = \mu(1 \otimes \chi)\psi,$$

relating χ to η, ϵ, μ and ψ, then $(H, \mu, \epsilon, \psi, \eta)$ is a *Hopf algebra*.

Remark. We have defined Hopf algebras to be commutative and cocommutative, because our examples will be.

Example 4.1.2. Let G be an abelian topological group, and E a spectrum with a Künneth theorem for the space G, i.e., an isomorphism $K : E_*(G) \otimes E_*(G) \to E_*(G \times G)$. Write

$\otimes = \otimes_{E_*}$. Then the group structure $\phi : G \times G \to G$, $\iota : G \to G$ along with the diagonal $\Delta : G \to G \times G$ induce maps

$$\begin{aligned} \mu &: E_*G \otimes E_*G \cong E_*(G \times G) \to E_*G && \text{(Multiplication)} \\ \psi &: E_*G \to E_*(G \times G) \cong E_*G \otimes E_*G && \text{(Comultiplication)} \\ \chi &: E_*G \to E_*G, && \text{(Conjugation)} \end{aligned}$$

where $\mu = \phi_* \circ K$, $\psi = K^{-1} \circ \Delta_*$ and $\chi = \iota_*$. These give E_*G the structure of a Hopf algebra. The classifying space BG of G, defined in Appendix B, is naturally a topological group so E_*BG is also a Hopf algebra as long as $E_*(BG) \otimes E_*(BG) \cong E_*(BG \times BG)$.

4.2 Products in Tor and Ext

In this section we develop the necessary theory of Tor and Ext over a graded (commutative) algebra R over a graded commutative ring K. We show that there are product maps which for a commutative augmented R imbue $A = \mathrm{Tor}^R_{*,*}(K, K)$ with the structure of an algebra. We also produce the Yoneda product in $B = \mathrm{Ext}^{*,*}_R(K, K)$ which is dual to a coproduct in A. When R is a commutative and cocommutative coalgebra, this imbues both A and B with the same structure. This material essentially follows [CE56, Chapter XI], where the reader may find proofs of most statements, with the only essential difference that our modules and algebras are graded, introducing a sign from the graded twist map

$$\tau : M \otimes N \to N \otimes M,$$

and giving a bigrading on Tor and Ext. We develop products in a slightly higher generality than we need. The next section is devoted to a chain level construction of the product in Tor, which is useful for computation. We also exhibit a procedure for computing Yoneda products.

Remark. By a chain complex of graded K-modules X, we a bigraded (left) K-module $X_{*,*}$ with components of the action $K_u \otimes X_{s,t} \to X_{s,t+u}$. The differential is given in bidegree (s, t) by a K-module map

$$X_{s,t} \to X_{s-1,t},$$

i.e., it is of bidegree $(-1, 0)$. Similarly for cochain complexes, but the bidegree of the differential is $(1, 0)$. If x is a homogeneous element in $X_{s,t}$, we call s the *filtration degree* of x, and t the *internal degree* of x. The sum $|x| := s + t$ is the *total degree* of x.

Remark. Recall that if M is a right K-module with module structure λ, and N is a left K-module with module structure λ', then $M \otimes_K N$ is is defined in the usual way as a coequalizer

$$M \otimes K \otimes N \underset{\lambda'}{\overset{\lambda}{\rightrightarrows}} M \otimes N,$$

and is given in degree t by

$$\bigoplus_{u+v=t} M_u \otimes_K N_v.$$

If however M is a left module and N is a right module, both are left modules, or both are right modules, the graded twist map introduces a sign.

Let K be a graded commutative ring and X, Y two chain complexes of graded K-modules. We define $X \otimes Y$ and the internal $\hom$-complex $\mathrm{Hom}_K(X, Y)$.

Definition 4.2.1. If X and Y are chain complexes of graded modules with differentials d' and d'', respectively, the chain complex $X \otimes_K Y$ in filtration degree s is given by the graded module

$$\bigoplus_{u+v=s} X_u \otimes_K Y_v$$

with differential $d_s = \sum_{u+v=s}(d'_u \otimes 1 + (-1)^u 1 \otimes d''_v)$.

Remark. Note that there is a natural map on homology

$$\alpha : H_{*,*}(X, d') \otimes H_{*,*}(Y, d'') \to H_{*,*}(X \otimes Y, d) \tag{4.1}$$

sending an element $[x] \otimes [y]$ to the class $[x \otimes y]$.

Notation. Given a graded K-module M denote by $\Sigma^u M$ the graded module given in degree t by $(\Sigma^u M)_t = M_{u-t}$.

Definition 4.2.2. Let again X and Y be two chain complexes of graded modules with differentials d and d', respectively. The *internal graded* $\hom$ *chain complex* $\mathrm{Hom}_K(X, Y)$ of graded K-modules is given in bidegree (s, t) by the K-module

$$\mathrm{Hom}_K^{s,t}(X, Y) = \prod_{l \in \mathbf{Z}} \mathrm{Hom}_K(X_{s+l}, \Sigma^{-t} Y_l),$$

which in (internal) degree t is the K-module of homomorphisms $\mathrm{Hom}_K(\Sigma^t X_s, Y_s)$. If $(f : X_{s+l} \to Y_l)$, $s \in \mathbf{Z}$ is of degree t, the differential is given by $d_s f = d'_s \circ f - (-1)^t f \circ d''_s$.

The algebra structure on $T = R \otimes S$. Recall that we write $\otimes = \otimes_K$.

Notation. For any pair of K-modules M and N let $\tau : M \otimes N \cong N \otimes M$ be the twist map sending $m \otimes n$ to $(-1)^{|m||n|} n \otimes m$.

Let R and S be two K-algebras with multiplication μ and μ', respectively. Then $T = R \otimes S$ is a graded K-algebra with multiplication

$$R \otimes S \otimes R \otimes S \xrightarrow{1 \otimes \tau \otimes 1} R \otimes R \otimes S \otimes S \xrightarrow{\mu \otimes \mu'} R \otimes S.$$

If M is a (left) R-module with module structure λ, and M' a (left) S-module with module structure λ', then $M \otimes M'$ is a (left) T-module with structure maps

$$R \otimes S \otimes M \otimes M' \xrightarrow{1 \otimes \tau \otimes 1} R \otimes M \otimes S \otimes M' \xrightarrow{\lambda \otimes \lambda'} M \otimes M'.$$

An external pairing in Tor. Let still M be a left R-module, and M' a left S-module. If X is an R-projective resolution M, and X' an S-projective resolution of M', then $X \otimes X'$ is a T-projective chain complex over the left T-module $M \otimes M'$. Hence, if Q is a right T-module, we get a K-module map

$$H(Q \otimes_T (X \otimes X')) \to \mathrm{Tor}^T_{*,*}(Q, M \otimes M'). \tag{4.2}$$

Let K be a *graded field*, meaning that every homogeneous generator of K is invertible. Then every K-module is free. We have the following two consequences:

- The T-projective complex $X \otimes X'$ is a T-projective *resolution* of $M \otimes M'$, so the map (4.2) is an isomorphism of K-modules.

- The map α of (4.1) is an isomorphism of K-modules.

Denote by ${}_RM$ that M is a left R-module, and by M_R that M is a right R-module.

Lemma 4.2.3. *Consider the following configuration of modules: $({}_RM\ , N_R, {}_SM'\ , N'_S)$. Then there is a natural external pairing of K-modules*

$$\top : \mathrm{Tor}^R_{*,*}(N, M) \otimes \mathrm{Tor}^S_{*,*}(N', M') \to \mathrm{Tor}^T_{*,*}(N \otimes N', M \otimes M').$$

Proof. Consider the map

$$\varphi_1 : (N \otimes_R M) \otimes (N' \otimes_S M') \to (N \otimes N') \otimes_T (M \otimes M')$$

given by $\phi((n \otimes m) \otimes (n' \otimes m')) = (-1)^{|m||n'|}(n \otimes n') \otimes (m \otimes m')$. Resolving M by an R-projective chain complex X and M' by an S-projective chain complex X', we get a map

$$\Phi_1 : (N \otimes_R X) \otimes (N' \otimes_S X') \to (N \otimes N') \otimes_T (X \otimes X')$$

Passing to homology and composing with (4.1) on the left we a K-module map

$$\mathrm{Tor}^R_{*,*}(N, M) \otimes \mathrm{Tor}^S_{*,*}(N', M') \to H((N \otimes N') \otimes_T (X \otimes X')),$$

which when composed with the map (4.2) gives the result. □

The algebra structure on $\mathrm{Tor}^R_{*,*}(K, K)$. Let now R be a commutative K-algebra. This means that the multiplication map $R \otimes R \to R$ is an R-algebra map. In other words we can consider R an algebra over itself. The above constructions, taking $T = R \otimes_R R$ and using that $R \otimes_R R \cong R$, give

$$\pitchfork\colon \mathrm{Tor}^R_{*,*}(N, M) \otimes \mathrm{Tor}^R_{*,*}(N', M') \to \mathrm{Tor}^R_{*,*}(N \otimes_R N', M \otimes_R M').$$

If R is augmented over K with augmentation $\epsilon : R \to K$, note that K is an R-module via ϵ. Further, let $M = M' = N = N' = K$ considered as an R-module. Using the isomorphism $K \otimes_R K \to K$ the above pairing becomes

$$\phi : \mathrm{Tor}^R_{*,*}(K, K) \otimes \mathrm{Tor}^R_{*,*}(K, K) \to \mathrm{Tor}^R_{*,*}(K, K).$$

Lemma 4.2.4. $\mathrm{Tor}^R_{*,*}(K, K)$ *with multiplication ϕ is an associative, graded commutative and unital K-algebra.*

The Yoneda product in $\mathrm{Ext}^{*;*}_R(K, K)$. There is also a product in $\mathrm{Ext}^{*;*}_R(K, K)$. Let M, N, and P be R-modules and consider the composition pairing

$$\mathrm{Hom}_R(N, P) \otimes \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M, P)$$

of R-chain complexes. Let Y be an R-injective resolution of P and X an R-projective resolution of M. Then we get a pairing

$$\mathrm{Hom}_R(N, Y) \otimes \mathrm{Hom}_R(X, N) \to \mathrm{Hom}_R(X, Y).$$

Taking homology and composing with α of (4.1) we get a map

$$\mathrm{Ext}^{*;*}_R(N, P) \otimes \mathrm{Ext}^{*;*}_R(M, N) \to \mathrm{Ext}^{*;*}_R(M, P),$$

the *Yoneda product* in Ext. When $M = N = P = K$ the Yoneda product gives $\mathrm{Ext}^{*;*}_R(M, P)$ the structure of a K-algebra.

Lemma 4.2.5. *Let R be a (commutative and cocommutative) Hopf algebra. Then the Yoneda product makes $\mathrm{Ext}_R^{*,*}(K,K)$ a commutative ring.*

Note. Adams, in his paper on elements of Hopf invariant 1, [Ada60], gives a procedure for computing Yoneda products in $\mathrm{Ext}_R^{*,*}(K,K)$ which we will employ in later sections. The next section is dedicated to computing products in $\mathrm{Tor}_{*,*}^R(K,K)$.

4.3 Computing products in Tor

In order to compute the product structure on $\mathrm{Tor}^R(K,K)$ where R is a graded commutative K-algebra, we pick a projective resolution B_* of K, and a map $(K \otimes B_*) \otimes (K \otimes B_*) \to (K \otimes B_*)$ which induces it. Let K be a graded field. Let R be an augmented K-algebra and $\overline{R} = \ker \epsilon$. We recall the *normalized bar resolution* $B_* = B_*(R,R,K)$ of K, which is an R-module via the augmentation. It is the chain complex associated to a simplicial bar construction modulo degeneracies. Less obtusely it has

$$B_k = R \otimes \overline{R}^{\otimes k} \otimes K$$

with elements $a_0[a_1|\cdots|a_k] = a$, $a_i \in \overline{R} = \ker(\epsilon)$ for $i = 1, \ldots, k$, and the boundary is

$$d(a) = a_0 a_1 [a_2|\cdots|a_k] + \sum_{i=1}^{k-1} (-1)^i a_0[\ldots|a_i a_{i+1}|\ldots].$$

(The $i = k$ term uses augmentation on $a_k \in \overline{R}$ and so vanishes).

Lemma 4.3.1. *The chain complex $B_* = B_*(R,R,K)$ is a resolution of K as a R-module, so the homology of $B_*(K,R,K) \cong K \otimes_R B_*(R,R,K)$ is $\mathrm{Tor}_{*,*}^R(K,K)$.*

Definition 4.3.2. The *shuffle product* on $(K \otimes_R B_*) \otimes (K \otimes_R B_*) \to (K \otimes_R B_*)$ sends

$$[a_1|\ldots|a_k] \otimes [a_{k+1}|\ldots|a_{k+l}]$$

to

$$\sum_{\sigma \in (k,l)\text{-shuffles}} \mathrm{sign}(\sigma)[a_{\sigma(1)}|\cdots|a_{\sigma(k+l)}].$$

Here a (k,l)-*shuffle* is a permutation which preserves the respective orderings of the first k and last l elements in $\{1, \ldots, k+l\}$, i.e.,

$$\sigma(1) < \cdots < \sigma(k) \quad \text{and} \quad \sigma(k+1) < \cdots < \sigma(k+l).$$

Applying the map $\alpha : H(X) \otimes H(Y) \to H(X \otimes Y)$ the shuffle product induces the product structure on Tor.

4.4 Tor over a truncated polynomial algebra

Let K be a graded field (meaning every homogenous generator is invertible) of characteristic $p \neq 2$.

Definition 4.4.1. Denote by $P_q(x) = K[x]/(x^q)$ a *truncated polynomial algebra* over K. It is a Hopf algebra with coproduct determined by x being primitive. We write

$$E(x) = P_2(x) = K[x]/(x^2)$$

for the *exterior algebra* on x.

Definition 4.4.2. Let $\Gamma(x) = \mathbf{F}_p\{\gamma_0 x, \gamma_1 x, \dots\}$ be the $\mathbf{F}_p$-vector space spanned by the symbols $\gamma_i x$, $i \geq 0$, and define a multiplication by $\gamma_i x \cdot \gamma_j x = \binom{i+j}{j}\gamma_{i+j}x$. Then $\gamma_0 x = 1$, and we write let $\gamma_1 x = x$. The algebra $\Gamma(x)$ is the *divided power algebra* on x. The coproduct

$$\psi(\gamma_k x) := \sum_{i+j=k} \gamma_i x \otimes \gamma_j x$$

and involution $\chi(x) = -x$ makes $\Gamma(x)$ a Hopf algebra.

This section serves as a proof of the following proposition.

Proposition 4.4.3. *Let $q = p^j$ and et $A = P_q(x)$ be a truncated polynomial K-algebra on x in even degree a. Then as a Hopf algebra*

$$\mathrm{Tor}^A_{*,*}(K, K) \cong E(\sigma x) \otimes \Gamma(\phi x)$$

where σx is represented by the bar cycle $[x]$ in bidegree $(1, a)$ and ϕx by the bar cycle $[x^{\frac{q}{p}(p-1)}|x^{\frac{q}{p}}]$ in bidegree $(2, aq)$ (in fact by any $[x^{r_1}|x^{r_2}]$ with $r_1 + r_2 = q$, $r_i > 0$), and $\sigma x, \phi x$ are the only primitive elements, up to units.

We first compute $\mathrm{Tor}^{P_q(x)}_{*,*}(K, K)$ as a K-module.

Lemma 4.4.4. *Recall that a is the degree of x. The K-module $\mathrm{Tor}^{P_q(x)}_{*,*}(K, K)$ is free on generators $p_i, i = 0, 1, \dots,$ with p_{2k} in bidegree $(2k, kaq)$, and p_{2k+1} in bidegree $(2k + 1, kaq + a)$.*

We show this by constructing a free resolution

$$K \xleftarrow{\epsilon} P_0 \xleftarrow{d_1} P_1 \xleftarrow{d_2} P_2 \xleftarrow{d_3} \cdots \qquad (P_i = A\{\text{generators}\}),$$

which will be of rank 1 in each degree and have the property that every free generator descends to a K-module generator of Tor.

Proof. Let $P_0 = A\{p_0\}$, with p_0 in degree $|p_0| = 0$, with $\epsilon(p_0) = 1$. The kernel of ϵ is generated by xp_0 in degree $|xp_0| = a$. Hence let $P_1 = A\{p_1\}$ with $|p_1| = a$ and $d_1 = x$ (multiplication by x). Then $d_1(x^{q-1}p_1) = x^q p_0 = 0$, so let $P_2 = A\{p_2\}$ with $|p_2| = a + (q-1)a$. In general, let

$$\begin{aligned} d_{2k} &= x^{q-1} & \qquad |p_{2k}| &= kaq \\ d_{2k+1} &= x & \qquad |p_{2k+1}| &= |p_{2k}| + a. \end{aligned}$$

It is easy to check that the resulting sequence is exact. Since $K \otimes_A d = 0$ this determines the additive structure of $\mathrm{Tor}^A_{*,*}(K, K)$ completely. □

We now turn to the algebra structure. The natural product on Tor of Section 4.2 allows us to identify generators in higher bidegrees in terms of $\sigma x := [p_1]$ and $\phi x := [p_2]$ (in bidegrees $(1, a)$ and $(2, qa)$ respectively).

Proof of the product structure. In order to compute products in Tor we compare our free resolution P_* to the bar resolution $B_* = B_*(A, A, K)$ of Section 4.3. The shuffle map $B_* \otimes B_* \to B_*$ from the proof of Proposition 4.4.3 computes products in Tor.

Observe that the generator $[x^{q-1}|x|\cdots|x^{q-1}|x]$ is a cycle in $K \otimes_A B_*$. We build our chain map as depicted in the diagram

$$\begin{tikzcd} P_0 \arrow[d, "{[]}"] & P_1 \arrow[l, "x"'] \arrow[d, "{[x]}"] & P_2 \arrow[l, "x^{q-1}"'] \arrow[d, "{[x^{q-1}|x]}"] & P_3 \arrow[l, "x"'] \arrow[d, "{[x|x^{q-1}|x]}"] & \cdots \arrow[l, "x^{q-1}"'] \\ B_0 & B_1 \arrow[l] & B_2 \arrow[l] & B_3 \arrow[l] & \cdots \arrow[l] \end{tikzcd} ,$$

where the lower horisontal arrows are given by

$$x[] \hookleftarrow [x], \quad x^{q-1}[x] - 0 \hookleftarrow [x^{q-1}|x], \quad \text{etc.},$$

and the vertical arrows are labelled by where they send the generator. In general the chain maps are given by

$$[x^{q-1}|x|\cdots|x^{q-1}|x] : P_{2k} \longrightarrow B_{2k}, \qquad [x|x^{q-1}|x|\cdots|x^{q-1}|x] : P_{2k+1} \longrightarrow B_{2k+1}$$

where $x^{q-1}|x$ is repeated k times.

From the proof of Proposition 4.4.3

$$[p_{2k}] \cdot [p_{2l}] = \binom{k+l}{l} [x^{q-1}|x|\cdots|x^{q-1}|x] + \lambda \qquad (x^{q-1}|x \text{ repeated } k+l \text{ times})$$

where λ consists of pairs of terms containing consecutive $x|x$ or $x^{q-1}|x^{q-1}$ which differ by the sign of a transposition and hence cancel, i.e., $\lambda = 0$. In conclusion,

$$[p_{2k}] \cdot [p_{2l}] = \binom{k+l}{l} [p_{2(k+l)}].$$

The products $[p_1] \cdot [p_{2l}]$ and $[p_1] \cdot [p_1]$ follow along the same lines:

$$\begin{aligned} [p_1] \cdot [p_{2l}] &= [x] \cdot [x^{q-1}|x| \cdots |x^{q-1}|x] \\ &= [x|x^{q-1}|x| \cdots |x^{q-1}|x] + \text{cancelling pairs} \\ &= [p_{2k+1}] \end{aligned}$$

and

$$[p_1] \cdot [p_1] = [x] \cdot [x] = [x|x] - [x|x] = 0.$$

Writing $\sigma x = [p_1] = [x]$ and $\phi x = [p_2] = [x^{q-1}|p]$ this allows us to express $\mathrm{Tor}^A_{*,*}(K, K)$ as $E(\sigma x) \otimes \Gamma(\phi x)$, which isright product structure. $\square$

Next we compute the coproduct on Tor. Since K is a graded field, $\mathrm{Ext}_A^{*,*}(K, K)$ is degreewise dual to $\mathrm{Tor}^A_{*,*}(K, K)$, with generators $[g_k]$ represented by

$$g_k := p_k^* : P_k \to \Sigma^{|p_k|} K.$$

Lemma 4.4.5. *As a bigraded K-algebra* $\mathrm{Ext}_A^{*,*}(K, K) \cong E([g_1]) \otimes P([g_2])$ *(where $P(x)$ is a polynomial K-algebra on x).*

Proof. We compute the Yoneda products $[g_1] \cdot [g_1]$, $[g_1] \cdot [g_{2k}]$ and $[g_{2k}] \cdot [g_{2l}]$.

$[g_1] \cdot [g_1] = 0$**:** In the diagram below we lift g_1 by a chain map. The shown composite $g_1 \circ x^{p-1}$ represents the product $[g_1] \cdot [g_1]$. This follows the methods in Adams [Ada60].

$$\begin{tikzcd}
K & P_0 \arrow[l] & P_1 \arrow[l] \arrow[dl, "g_1"'] \arrow[d, "1"] & P_2 \arrow[l] \arrow[d, "x^{q-1}"'] \arrow[dd, bend left, "g_1 \circ x^{q-1}"] \\
& K & P_0 \arrow[l, "\epsilon"] & P_1 \arrow[l] \arrow[d, "g_1"'] \\
& & & K.
\end{tikzcd}$$

Since $g_1 \circ x^{q-1}$ is identically zero, $[g_1] \cdot [g_1] = 0$.

$[g_1]\cdot[g_{2k}]=[g_{2k+1}]$**:** Using the same method as above, we lift g_{2k} by a chain map. The composite $g_1\circ 1 = g_{2k+1}$ represents $[g_1]\cdot[g_{2k}]$.

$$\begin{tikzcd}
 & P_{2k} \arrow[dl, "g_{2k}"'] \arrow[d, "1"] & P_{2k+1} \arrow[l, "x"'] \arrow[d, "1"'] \arrow[dd, bend left, "g_1\circ 1 = g_{2k+1}"] \\
K & P_0 \arrow[l] & P_1 \arrow[l, "x"'] \arrow[d, "g_1"'] \\
 & & K.
\end{tikzcd}$$

The composition shown is equal to g_{2k+1} which represents $[g_{2k+1}]$.

$[g_{2k}]\cdot[g_{2l}]=[g_{2(k+l)}]$**:** We lift g_{2k} by a chain map and compose the result with g_{2l}, as above, to get a representative for $[g_{2k}]\cdot[g_{2l}]$.

$$\begin{tikzcd}
 & P_{2k} \arrow[dl, "g_{2k}"'] \arrow[d, "1"] & P_{2k+1} \arrow[l, "x"'] \arrow[d, "1"'] & \cdots \arrow[l] & P_{2(k+l)} \arrow[l, "x^{q-1}"'] \arrow[d, "1"'] \arrow[dd, bend left, "g_{2l}\circ 1 = g_{2(k+l)}"] \\
K & P_0 \arrow[l] & P_1 \arrow[l] & \cdots \arrow[l] & P_{2l} \arrow[l] \arrow[d, "g_{2l}"'] \\
 & & & & K.
\end{tikzcd}$$

Since $g_{2l}\circ 1 = g_{2(k+l)}$, $[g_{2k}]\cdot[g_{2l}]=[g_{2(k+l)}]$. This proves the lemma. $\square$

Lemma 4.4.6. *In the free K-module $\mathrm{Tor}^{P_q(x)}(K,K)$ with the basis computed above, and coalgebra structure dual to the Yoneda product, the only primitive elements (up to scalars) are $\sigma x := [p_1]$ and $\phi x := [p_2]$.*

Remark. This lemma should follow from generalities on Hopf algebras and have an easier proof than what follows. Even so, we only use duality of K-linear maps and the previous lemma.

Proof. We write $\Delta^* : \mathrm{Ext}\otimes\mathrm{Ext}\to\mathrm{Ext}$ for the Yoneda product, which is dual to the product $\Delta : \mathrm{Tor}\to\mathrm{Tor}\otimes\mathrm{Tor}$ int Tor. Then, by duality of linear maps

$$\Delta(p_n)=\sum_{k+l=n}\Delta^*(g_k\otimes g_l)(p_n)p_k\otimes p_l.$$

This means that, omitting the square brackets around $[g_i]$ and $[p_i]$ for easier reading,

$$\begin{aligned}\Delta(\sigma x) = \Delta(p_1) &= \Delta^*(g_1 \otimes g_0)(p_1) p_1 \otimes p_0 + \Delta^*(g_0 \otimes g_1)(p_1) p_0 \otimes p_1 \\ &= p_1 \otimes p_0 + p_0 \otimes p_1 = \sigma x \otimes 1 + 1 \otimes \sigma x\end{aligned}$$

is a primitive. This uses that g_1 is the multiplicative unit in Ext. Next we compute $\Delta(\phi x)$:

$$\begin{aligned}\Delta(\phi x) = \Delta(p_2) = &\Delta^*(g_2 \otimes g_0)(p_2) p_2 \otimes p_0 \\ &+ \Delta^*(g_1 \otimes g_1)(p_2) p_1 \otimes p_1 \quad (= 0) \\ &+ \Delta^*(g_0 \otimes g_2)(p_2) p_0 \otimes p_2.\end{aligned}$$

The generator p_{2k} is not primitive for $k > 1$, because $\Delta(p_{2k})$ has the non-zero term $\Delta^*(g_{2k-2} \otimes g_2)(p_{2k}) p_{2k-2} \otimes p_2$. A similar argument holds in the odd case p_{2k+1}, $k > 0$. $\square$

5 The bar spectral sequence

The idea of a spectral sequence was first introduced by Leray as a tool to compute sheaf cohomology. In Serre's thesis [Ser51] the tool was sharpened and applied to compute unstable homotopy groups of spheres. The useful reference for both generalities and applications of spectral sequences in algebraic topology is McCleary [McC01]. Conditional and strong convergence of spectral sequences is discussed in an article by Boardman [Boa99]. For an introduction to spectral sequences, Bott-Tu [BT82] develops spectral sequences as a generalization of the Mayer-Vietoris sequence with applications to unstable homotopy groups of spheres. An exposition of motivation and generalities may be found in unpublished notes by Rognes [Rog15a], and the addendum [Rog15b] where the Adams spectral sequence is in detailed computations of stable homotopy groups of spheres. We first discuss spectral sequences in general, then conditional and strong convergence followed by a discussion on products and coproducts and their convergence to structure on the abutment. We then develop the bar spectral sequence, discuss differential structure in a Hopf algebra spectral sequences and compute a detailed example.

5.1 The spectral sequence associated to an unrolled exact couple

A *spectral sequence* is on the face of it nothing but a sequence $(E^r, d^r), r = 1, 2, \dots,$ of differential abelian groups for which $H(E^r, d^r) \cong E^{r+1}$. We call E^r the E^r-*page* or *term* of (E^r, d^r). As one adds structure such as a bigrading or multiplicative structure on each page, these become rather complicated objects. They arise, however, quite naturally in attempting to compute the homology E_*X (or cohomology E^*X) of a filtered space

$$\cdots \subset X_{s-1} \subset X_s \subset \cdots \subset X$$

from the relative homologies $E_*(X_s, X_{s-1}), s \in \mathbf{Z}$, or what amounts to the reduced homology of the filtration coefficients X_s/X_{s-1}. Note that applying E_* induces a diagram

$$\begin{tikzcd}
\dots \arrow[r, "i_{s-1}"] & E_*X_{s-1} \arrow[r, "i_s"] \arrow[d, "j_{s-1}"] & E_*X_s \arrow[r, "i_{s+1}"] \arrow[d, "j_s"] & E_*X_{s+1} \arrow[r, "i_{s+2}"] \arrow[d, "j_{s+1}"] & \dots \\
\dots & E_*(X_{s-1}, X_{s-2}) \arrow[ul, "\partial_{s-1}"] & E_*(X_s, X_{s-1}) \arrow[ul, "\partial_s"] & E_*(X_{s+1}, X_s) \arrow[ul, "\partial_{s+1}"] & \dots \arrow[ul, "\partial_{s+2}"]
\end{tikzcd}$$

where ∂ is of degree -1 and each triangle (i_s, j_s, ∂_s) is exact. Writing

$$E^1_{s,t} = E_{s+t}(X_s, X_{s-1}) \text{ and } d^1_{s,t} = (j_{s-1} \circ \partial_s)_{s+t}$$

we get a chain complex of abelian groups (E^1, d^1). The homology of this complex is a new abelian group E^2 which inherits a differential from the diagram above, which we shall analyze in the general situation.

Definition 5.1.1. An *unrolled exact couple* is a family of tuples $(A_s, E_s, i_s, j_s, k_s), s \in \mathbf{Z}$ where A_s, E_s are abelian groups together with homomorphisms $i_s : A_s \to A_{s+1}$, $j : A_s \to E_s$, $k : E_s \to A_{s-1}$ making the diagram

$$\begin{tikzcd}
A_{s-1} \arrow[rr, "i_{s-1}"] & & A_s \arrow[dl, "j_s"] \\
& E_s \arrow[ul, "k_s"] &
\end{tikzcd}$$

exact at each vertex.

Given an unrolled exact couple as above, the map $d_* = j_{*-1} \circ k_*$ is a differential on E_* with homology $E'_* := H(E_*, d_*)$. Let $A'_s = \operatorname{im} i_{s-1}$ and define $i'_s(i_{s-1}x) = i_s(i_{s-1}x)$, $j'_s(i_{s-1}x) = [j(x)]$, and $k'([y]) = k(y)$. The homomorphisms i'_s, j'_s, k'_s are well defined, and

$$(A'_s, E'_s, i'_s, j'_s, k'_s), s \in \mathbf{Z}$$

is an unrolled exact couple, the *derived unrolled exact couple* of $(A_*, E_*, i_*, j_*, k_*)$.

Writing $(A^{(r)}_*, E^{(r)}_*, i^{(r)}_*, j^{(r)}_*, k^{(r)}_*)$ for the $(r-1)$-fold iteration of this construction, i.e.,

$$(A_*, E_*, i_*, j_*, k_*) = (A^{(1)}_*, E^{(1)}_*, i^{(1)}_*, j^{(1)}_*, k^{(1)}_*),$$

we get a spectral sequence (E^r_*, d^r_*) with $E^r_* = E^{(r)}_*$ and $d^r_* = j^{(r)}_* \circ k^{(r)}_*$.

A direct description of this spectral sequence possible. Let $(A_s, E_s, i_s, j_s, k_s), s \in \mathbf{Z}$ be an unrolled exact couple and write

$$i^r_s = i_{s-1} \circ \cdots \circ i_{s-r} : A_{s-r} \to A_s \quad (r > 0).$$

Define the r-*cycles* $Z_s^r = k^{-1}\operatorname{im}(i_{s-1}^{r-1})$ and r-*boundaries* $B_s^r = j(\ker(i_{s+r-1}^{r-1}))$. Then $B_s^1 = 0$ and $Z_s^1 = E_s$, and

$$B_s^{r-1} \subset B_s^r \subset Z_s^{r'} \subset Z_s^{r'+1}$$

for all $r, r' > 0$. Further, $E_s^r \cong Z_s^r/B_s^r$ and $d_s^r([x]) = [j_{s-r}(y)]$ where $x \in Z_s^r$ and $k_s(x) = i_{s-1}^{r-1}(y)$.

Remark. The example above corresponds to $A_{s,t} = E_{s+t}X_s$ and $E_{s,t} = E_{s+t}(X_s, X_{s-1})$ with i, j, k having bidegrees $(1,0)$, $(0,1)$, $(-1,-1)$. Then $d_{s,t}^r$ has bidegree $(-r, r-1)$.

5.2 Strong convergence

Let $(A_*^{(r)}, E_*^{(r)}, i_*^{(r)}, j_*^{(r)}, k_*^{(r)})$ be an unrolled exact sequence and define the *infinite cycles*

$$Z_s^\infty = \bigcap_r Z_s^r$$

and *infinite boundaries*

$$B_s^\infty = \bigcup_r B_s^r.$$

The E^∞-*term* of the associated spectral sequence is $E_s^\infty := Z_s^\infty/B_s^\infty$.

Definition 5.2.1. We say that E^r *converges strongly* or *abuts* to a group G, and write $E_*^r \Rightarrow G$, if there is a filtration

$$\cdots \subset F_{s-1} \subset F_s \subset F_{s+1} \subset \cdots \subset G$$

of G such that $\overline{F}_s = F_s/F_{s-1} \cong E_s^\infty$.

Define $A_\infty = \operatorname{colim} A_s$ and $A_{-\infty} = \lim_s A_s$.

Theorem 5.2.2 ([Boa99]). *If $E_s = 0$ and $A_s \cong A_{-\infty}$ for all $s < 0$, then the spectral sequence (E^r, d^r) converges strongly to A_∞ with filtration $F_sA_\infty = \operatorname{im}(A_s \to A_\infty)$.*

5.3 Multiplicative structure and convergence

Let $\{'E^r\}$, $\{''E^r\}$ and $\{E^r\}$ be (bigraded) spectral sequences, and write $|x|$ for the total degree of an element x. A *pairing of spectral sequences* is a sequence

$$\{\phi^r : {'E^r} \otimes {''E^r} \to E^r\}_{r>0}$$

of pairings of bilinear maps such that

$$d^r(\phi^r(x \otimes y)) = \phi^r('d^r\ (x) \otimes y) + (-1)^{|x|}\phi^r(x \otimes {''d^r}\ (y))$$

and the diagram

$$\begin{tikzcd}
H('E^r) \otimes H(''E^r) \arrow[r] \arrow[d, "\cong"'] & H('E^r \otimes ''E^r) \arrow[r, "\phi^r_*"] & H(E^r) \arrow[d, "\cong"] \\
'E^{r+1} \otimes ''E^{r+1} \arrow[rr, "\phi^{r+1}"'] & & E^{r+1}
\end{tikzcd}$$

commutes, meaning each ϕ_r is compatible with d^r and that ϕ_{r+1} is the pairing induced by ϕ_r.

Assume the natural map $\alpha : H('E^r) \otimes H(E^r) \to H('E^r \otimes E^r)$ is an isomorphism. Then we similarly define a *copairing* as a sequence $\{\psi^r : {'E^r} \to {''E^r} \otimes E^r\}_{r>1}$ of copairings such that

$$d^r(\psi^r(x)) = ('d^r - {''d^r})\psi^r(x)$$

and the diagram

$$\begin{tikzcd}
H('E^r) \arrow[r, "\psi^r_*"] \arrow[d, "\cong"'] & H(''E^r \otimes E^r) \arrow[r, "\alpha^{-1}"] & H(''E^r) \otimes H(E^r) \arrow[d, "\cong"] \\
'E^{r+1} \arrow[rr, "\psi^{r+1}"'] & & ''E^{r+1} \otimes E^{r+1}
\end{tikzcd}$$

commutes. If $\{E^r\} = \{'E^r\} = \{''E^r\}$ a pairing or copairing is called a product or coproduct, respectively, and may be associative, coassociative, commutative, cocommutative, etc., in the usual way.

If now $\{'E^r\}$, $\{''E^r\}$ and $\{E^r\}$ abut to $'G$, $''G$ and G, and $\mu : {'G} \otimes {''G} \to G$ is a pairing of groups which is compatible with the filtrations of $'G$, $''G$, G in the sense that it restricts to a pairing

$$'F_u \otimes {''F_v} \to F_{u+v},$$

then a pairing ϕ as above *converges to* μ if the diagram

$$\begin{tikzcd}
'E^\infty \otimes {''E^\infty} \arrow[r, "\phi"] \arrow[d] & E^\infty \arrow[d] \\
\mathrm{gr}\, 'G \otimes \mathrm{gr}\, ''G \arrow[r, "\mu"'] & \mathrm{gr}\, G,
\end{tikzcd}$$

commutes, and similarly for copairings.

Definition 5.3.1. A *Hopf algebra spectral sequence* is a spectral sequence $\{E^r\}$ together with a Hopf algebra structure each E^rcompatible with d^r and, the Hopf algebra structure on E^{r+1} is the one induced on the homology of E^r via the identification

$$E^{r+1} \cong H(E^r, d^r).$$

5.4 The bar spectral sequence

The spectral sequence of a filtered space was first discovered by Milnor [Mil56], and the E^2-page was identified by Moore. The bar spectral sequence was later fully developed for H-spaces by Rothenberg-Steenrod [RS65], where they also show that it is a Hopf algebra spectral sequence. In this section we develop the bar spectral sequence of an abelian topological group. It computes generalized homology of the classifying space the group from Tor over the generalized homology of the group.

Let G be an (abelian) topological group, and E a spectrum with a Künneth theorem for the spaces involved. Consider the skeletal filtration

$$BG_0 \subset \cdots \subset BG_s \subset \cdots \subset BG = |[n] \mapsto G^{\times n}|.$$

Recall that each filtration quotient $B_sG/B_{s-1}G \cong S^s \wedge G^{\wedge s}$ as described in Appendix B. Applying $E_*(-)$ and the Künneth isomorphism we get

$$E^1_{s,t}(G) \cong \widetilde{E}_{s+t}(B_sG/B_{s-1}G) \cong \widetilde{E}_{s+t}(S^s \wedge G^{\wedge s}) \cong (\widetilde{E}_*(G) \otimes \cdots \otimes \widetilde{E}_*(G))_t,$$

where we have written $\otimes = \otimes_{E_*}$.

Definition 5.4.1. Let E be a spectrum with Künneth theorem for a topological abelian group G. The *bar spectral sequence* is the spectral sequence associated to the bar filtration of BG after the identification $\widetilde{E}_{s+t}(S^s \wedge G^{\wedge s}) \cong (\widetilde{E}_*(G) \otimes \cdots \otimes \widetilde{E}_*(G))_t$,

It may be shown that the d^1-differential is $\sum(-1)^i d_i$, see [May72, Theorem 11.13]. When E_* is a graded field, this is the bar complex which computes $\mathrm{Tor}^{E_*G}_{*,*}(E_*, E_*)$. Then the bar spectral sequence

$$\mathrm{Tor}^{E_*G}_{*,*}(E_*, E_*) \Longrightarrow E_*BG$$

converges to E_*BG. Part of the power and computability of the bar spectral sequence is that Tor over a graded field possesses a natural Hopf algebra structure which, in our case, converges to the Hopf algebra structure of E_*BG. We summarize in a theorem.

Theorem 5.4.2. *Let G be an abelian topological group and $E_*(-)$ a homology theory with Künneth theorem for G. Then the bar spectral sequence*

$$E^r_{*,*}(G) \Rightarrow E_*(BG)$$

converges strongly. Furthermore, the diagonal $\Delta : G \to G \times G$, product $\mu : G \times G \to G$ and inverse $G \to G$ gives the spectral sequence the structure of a Hopf algebra spectral sequence which converges to the corresponding structure on $E_*(G)$.

For proof of this fact see [RS65]. It essentially follows from the fact that the group structure and diagonal on BG are compatible with the filtration on BG, see Appendix B, this induces a Hopf algebra structure on the E^2-page which converges to the structure in homology induced by the product and diagonal of G.

5.5 Differentials in Hopf algebra spectral sequences

We recall here two key lemmas for determining the differential structure in the spectral sequences we encounter. They are also to be found in Ravenel-Wilson [RW80, Section 6].

Notation. We write $a \doteq b$ if $a = ub$ for some unit $u \in K$. Note that since K is graded, it does not follow from $a \doteq b$ that a and b lie in the same degree.

Lemma 5.5.1 (Lemma 6.9 [RW80])**.** *If $E(x) \otimes \Gamma(y)$ is a differential Hopf algebra with differential determined by $d(\gamma_{p^k}(y)) \doteq x$, then the homology is the sub-Hopf algebra of $\Gamma(y)$ generated by $\gamma_i(y)$, $0 \leq i < p^k$.*

Proof. For degree reasons $\gamma_i(y)$ is a cycle when $0 \leq i < p^k$. We show inductively on n that $d(\gamma_n(y)) = x\gamma_{n-p^k}(y)$. Since d is a differential of coalgebras we have $(d \otimes 1 + 1 \otimes d)\psi = \psi d$. Inductively we get

$$\begin{aligned}(d \otimes 1 + 1 \otimes d)\psi(\gamma_n(y)) =& (d \otimes 1 + 1 \otimes d) \sum_i \gamma_{n-i} y \otimes \gamma_i y \\ =& d\gamma_n y \otimes 1 + 1 \otimes d\gamma_n y \\ &+ \sum_i x\gamma_{n-i-p^k} y \otimes \gamma_i y + \sum_i \gamma_{n-i} y \otimes x\gamma_{i-p^k} y.\end{aligned}$$

Then we must have $d\gamma_n y = x\gamma_{n-p^k} y$ to get the appropriate coproduct. $\square$

Lemma 5.5.2 (Lemma 6.10 [RW80])**.** *Let $A = \bigotimes_i P_{(k_i)}(x_i)$. Define $y_i = \pm x_i^{p^{k_i}-1}$ so that*

$$\mathrm{Tor}^A_{*,*}(K, K) \cong \bigotimes_i E(\sigma x_i) \otimes \Gamma(\phi y_i),$$

and assume that the degrees of the x_i are distinct, and that the degrees of the y_i are distinct. Then

(a) The only possible non-trivial differentials are of the form

$$d^{2p^{n_i}-1}\gamma_{p^{n_i}}(\phi y_i) \doteq \sigma x_j$$

for some i, j and $n_i > 0$.

(b) If there is a permutation τ, and n_i's such that

$$d^{2p^{n_i}-1}\gamma_{p^{n_i}}(\phi y_i) \doteq \sigma x_{\tau(i)}$$

for all i, then the E^∞ term of the spectral sequence is the subalgebra of $\operatorname{Tor}^A_{*,*}(K, K)$ generated by $\gamma_{p^j}\phi y_i$ for $0 \leq j < n$.

Proof. Part (b) follows from Lemma 5.5.1 and part (a). An element of lowest degree supporting a non-trivial differential differential must be a generator, and the target of the differential must be primitive. For degree reasons σx_j and ϕy_i are permanent cycles, so the first non-trivial differential must be on $\gamma_{p^{n_i}}\phi y_i$ for some y_i and $n_i > 0$. This has even degree, so its target must be a primitive of odd degree, i.e., one of the σx_j. Lemma 5.5.1 computes the homology, and since the degrees of the σx_j are distinct this is well defined, and the homology is a sub-Hopf algebra as in Lemma 5.5.1, which supports no differentials for degree reasons, tensored with the kind of algebra we started with. Repeating this process yields the result. □

5.6 Example: The bar spectral sequence converging to $H_*(B^3\mathbf{Z}; \mathbf{F}_p)$

In order to illustrate the power of the bar spectral sequence in conjunction with the Hopf algebra structure on Tor we compute the spectral sequence converging to E_*BG for the case where $E = H = H\mathbf{F}_p$ with $p > 2$, and G a topological group model for $\mathbf{C}P^\infty = B^2\mathbf{Z}$ with $\mu : \mathbf{C}P^\infty \times \mathbf{C}P^\infty \to \mathbf{C}P^\infty$ classifying tensor product and $B\mathbf{C}P^\infty = B^3\mathbf{Z}$.

Let π be a finitely generated abelian group. For odd p computation of the mod p cohomology of $B^n\pi$ was first done by Cartan [Car54]. For $p = 2$ the computation of $H^*(B^n\pi, \mathbf{F}_p)$ is due to Serre [Ser53]. Using the bar spectral sequence we compute here, for odd p, mod p homology of $B^3\mathbf{Z}$, up to multiplicative extension problems. We do this without invoking the Steenrod algebra, relying instead on the algebraic properties of Hopf algebras. This makes these methods amenable to the study of generalized homology and cohomology. In order to compute multiplicative extensions one must essentially do as in [Wil82] to compute $H_*B^2\mathbf{Z}/(p^j)$, and use the isomorphism

$$H_*B^3\mathbf{Z} \cong \operatorname{colim}_j H_*B^2\mathbf{Z}/(p^j),$$

which is the approach for K_2 in the next chapter.

Remark. In fact the brunt of the work necessary to compute $H_*B^k\mathbf{Z}/(p^j)$ for any k is done in [Wil82], and in general

$$H_*B^{k+1}\mathbf{Z} \cong \operatorname{colim}_j H_*B^k\mathbf{Z}/(p^j).$$

Recall that as a Hopf algebra $H^*\mathbf{C}P^\infty = \mathbf{F}_p[\![x]\!]$, where $|x| = 2$ and

$$\psi(x) = x \otimes 1 + 1 \otimes x.$$

The augmentation sends x to $0 \in \mathbf{F}_p$.

Lemma 5.6.1. *The homology $H_*\mathbf{C}P^\infty \cong (H^*\mathbf{C}P^\infty)^*$ is $\mathbf{F}_p$-free on generators β_i dual to x^i. Furthermore the induced product in homology is given by $\beta_i \cdot \beta_j = \binom{i+j}{j}\beta_{i+j}$. This means that, writing $\beta = \beta_1$, we have an isomorphism $H_*\mathbf{C}P^\infty \cong \Gamma(\beta)$ as Hopf algebras (where $\gamma_i\beta = \beta_i$).*

Proof. The coefficient of $\beta_i \cdot \beta_j$ as a multiple of β_{i+1} is determined by duality and the Hopf algebra structure on $H^*\mathbf{C}P^\infty$:

$$\langle \beta_i \cdot \beta_j, x^k, \rangle = \langle \beta_i \otimes \beta_j, \psi(x^k)\rangle,$$

where

$$\begin{aligned}\psi(x^k) = (\psi(x))^k &= (x \otimes 1 + 1 \otimes x)^k \qquad \text{(product on } H^*\mathbf{C}P^\infty \otimes H^*\mathbf{C}P^\infty\text{)}\\ &= \sum_{l=0}^{k} \binom{k}{l} x^{k-l} \otimes x^l.\end{aligned}$$

Taking $k = i + j$ we get

$$\langle \beta_i \otimes \beta_j, \sum_{l=0}^{i+j} \binom{i+j}{l} x^{i+j-l} \otimes x^l\rangle = \binom{i+j}{j},$$

so $\beta_i \cdot \beta_j = \binom{i+j}{j}\beta_{i+j}$. $\square$

Notation. To make expressions easier to read we will often write $(i) := p^i$ in what follows.

Proposition 5.6.2. *The bar spectral sequence $E^2_{*,*} = \operatorname{Tor}^{H_*B^2\mathbf{Z}}(\mathbf{F}_p, \mathbf{F}_p) \Rightarrow H_*B^3\mathbf{Z}$ collapses at the E^2-page, and $E^2_{*,*} = E^\infty_{*,*} = \bigotimes_{i\geq 0} E(\sigma(\gamma_{(i)}\beta)) \otimes \Gamma(\phi(\gamma_{(i)}\beta))$ as a Hopf algebra, where $\gamma_{(i)}\beta = \gamma_{p^i}\beta$.*

We prove the above proposition in the remainder of the section.

Theorem 5.6.3. *A divided power algebra $\Gamma(x)$ over $\mathbf{F}_p$ has an algebra decomposition $\Gamma(x) = \bigotimes_{i\geq 0} P_p(\gamma_{(i)}x)$ where $P_p(y) = \mathbf{F}_p[y]/(y^p)$.*

Proof. This is proven in Milnor-Moore [MM65] or McCleary [McC01, Lemma 7.26]. □

By Theorem 5.6.3 the E^2-page of our spectral sequence decomposes as

$$E^2 = \operatorname{Tor}^{\Gamma(\beta)}_{*,*}(\mathbf{F}_p, \mathbf{F}_p) = \bigotimes_{i\geq 0} \operatorname{Tor}^{P_p(\gamma_{(i)}\beta)}_{*,*}(\mathbf{F}_p, \mathbf{F}_p) \Longrightarrow H_*B^3\mathbf{Z}.$$

The following lemma is a direct corollary of Lemma 4.4.3.

Lemma 5.6.4. *Let $A = P_p(x)$ with x in even degree $|x| = a$ and augmentation $\epsilon : x \mapsto 0$. As a Hopf algebra $\operatorname{Tor}^{P_p(\gamma_{(i)}\beta)}_{*,*}(\mathbf{F}_p, \mathbf{F}_p)$ is isomorphic to*

$$E(\sigma\gamma_{(i)}\beta) \otimes \Gamma(\phi\gamma_{(i)}\beta).$$

We are now in position to determine the differentials in the bar spectral sequence converging to $H_*B^3\mathbf{Z}$. The full E^2-page takes the form

$$\begin{aligned} \operatorname{Tor}^{\Gamma(\beta)}_{*,*}(K, K) &\cong \bigotimes_{i\geq 0} \operatorname{Tor}^{A_i}_{*,*}(K, K) \qquad\qquad A_i = P_p(\gamma_{(i)}\beta) \\ &\cong \bigotimes_{i\geq 0} E(\sigma\gamma_{(i)}\beta) \otimes \Gamma(\phi\gamma_{(i)}\beta). \end{aligned}$$

Proposition 5.6.5. *This spectral sequence collapses at the E^2-page.*

Proof. We show this by appealing to the Hopf algebra structure on Tor. Recall that the differentials are compatible with the Hopf algebra structure. Therefore it is enough to specify differentials on indecomposable elements (algebra generators). Furthermore the first non-trivial differential on an indecomposable must hit a primitive element.

The indecomposable elements in our algebra and their respective bidegrees are

$$\begin{aligned} \sigma\gamma_{(i)}\beta &\quad (1, 2p^i) \\ \gamma_{(0)}\phi\gamma_{(i)}\beta = \phi\gamma_{(i)}\beta &\quad (2, 2p^{i+1}) \\ \gamma_{(j)}\phi\gamma_{(i)}\beta &\quad (2p^j, 2p^{i+j+1}) \quad (j > 0) \end{aligned}$$

where the first two rows are all the primitive elements (up to scaling).

The indecomposable $\sigma\gamma_{(i)}$ does not support a non-trivial differential, as any d_r for $r > 1$ must leave the first quadrant.

Recall that we write $a \doteq b$ if a is a unit multiple of b. Since both $\gamma_{(i)}\phi\gamma_{(j)}\beta$ and $\phi\gamma_{(k)}\beta$ are of even total degree and d_r lowers total degree by 1, the relation

$$d_r(\gamma_{(i)}\phi\gamma_{(j)}\beta) \doteq \phi\gamma_{(k)}\beta$$

cannot occur.

The final case to check is whether

$$d_r(\gamma_{(j)}\phi\gamma_{(i)}\beta) \doteq \sigma\gamma_{(k)}\beta.$$

As the bidegree of d_r is $(-r, r-1)$, for this to occur we must have

$$(2p^j - r, 2p^{i+j+1} + r - 1) = (1, 2p^k).$$

Thus $r = 2p^j - 1$ and $r = 2p^k - 2p^{i+j+1} + 1$, but then

$$2p^j = 2p^k - 2p^{i+j+1} + 2,$$

and since $j, k, i + j + 1 > 0$ this implies the contradiction

$$0 \equiv 2 \pmod{p > 2}.$$

Hence the spectral sequence collapses at the E^2-page. $\square$

6 K_2-homology of $B^3\mathbf{Z}$

In this section we compute $K_{2*}B^3\mathbf{Z}$ as an algebra. This result is a special case of a more general result by Ravenel-Wilson [RW80] for the Morava K-theory of $K(\mathbf{Z}, k)$ for every k. Little to nothing new happens in the case of K_2 compared to the case of $K(2)$. Still we hope writing out our case in detail helps to illustrate their methods in the simplest case that still has most of the complexity of the full result, but with considerably fewer indices. Our main interest in this, however, is the fact that the algebra structure on $K_{2*}B^3\mathbf{Z}$ is dual to the coalgebra structure on $K_2^*B^3\mathbf{Z}$, which allows us to compute the set of group-like elements in K_2-cohomology. These are represented by homomorphisms of symmetric ring spectra from the suspension spectrum of $B^3\mathbf{Z}$ to K_2.

We will write K for K_2 as defined in Chapter 3 for an odd prime p, with $K_* = \mathbf{F}_{p^2}[u, u^{-1}]$ with $|u| = 2$. This section serves as a proof of the following result.

Theorem 6.0.6. *As algebras, there is an isomorphism*

$$K_*B^3\mathbf{Z} \cong \bigotimes_{k\geq 0} K_{2*}[\beta_{(2k,1)}]/(\beta_{(2k,1)}^p + \beta_{(2k,1)}),$$

where the $\beta_{(2k,1)}$ are defined below.

Let $\alpha : \mathbf{Z}/(p^j) \to \mathbf{Z}/(p^{j+1})$ denote the standard inclusion. By abuse of notation we write α also for the induced map $B^k\mathbf{Z}/(p^j) \to B^k\mathbf{Z}/(p^{j+1})$ of iterated bar constructions. As a rough outline we first note that, using these maps, the E^2-page of

$$\mathrm{Tor}^{K_*B^2\mathbf{Z}}(K_*, K_*) \Rightarrow K_*B^3\mathbf{Z}$$

may be presented as a colimit over j of $\mathrm{Tor}^{K_*B\mathbf{Z}/(p^j)}(K_*, K_*)$, so we compute

$$\mathrm{Tor}^{K_*B\mathbf{Z}/(p^j)}(K_*, K_*) \Rightarrow K_*B^2\mathbf{Z}/(p^j)$$

for each $j \geq 1$. In order to determine multiplicative extensions in this spectral sequence we label the elements of $E^2_{*,*}(B\mathbf{Z}/(p^j))$ using a pairing

$$\circ : E^2_{*,*}(\mathbf{Z}/(p^j)) \otimes K_*B\mathbf{Z}/(p^j) \to E^2_{*,*}(B\mathbf{Z}/(p^j))$$

converging to the *circle product*

$$\circ : K_*B\mathbf{Z}/(p^j) \otimes K_*B\mathbf{Z}/(p^j) \to K_*B^2\mathbf{Z}/(p^j)$$

induced by the cup product, where $E^r_{*,*}(G)$ denotes the bar spectral sequence converging to K_*BG. This product behaves well with respect to the Frobenius and Verschiebung maps on the Hopf algebras involved, which allows us to solve all multiplicative extension problems in

$$E^2_{*,*}(B\mathbf{Z}/(p^j)) \Rightarrow K_*B^2\mathbf{Z}/(p^j),$$

computing $K_*B^2\mathbf{Z}/(p^j)$ as an algebra. From there we compute the maps in the filtered system

$$\alpha_* : K_*B^2\mathbf{Z}/(p^j) \to K_*B^2\mathbf{Z}/(p^{j+1}),$$

and then $K_*B^3\mathbf{Z}$ is a colimit of the $K_*B^2\mathbf{Z}/(p^j)$ as j increases.

6.1 Some preliminary computations

First some results on the coalgebra and algebra structure on $K_*B^2\mathbf{Z}$. Recall that K is a complex oriented theory, see Chapter 3 for definition, and so $K^*B^2\mathbf{Z} = K^*\mathbf{C}P^\infty$ is a formal power series ring on a generator x in degree 2. As an $\mathbf{F}_p$-vector space $K_*B^2\mathbf{Z}$ is free on β_i dual to x^i, $i = 0, 1, \ldots$, and the coalgebra structure is given by

$$\psi(\beta_k) = \sum_{i=0}^{k} \beta_{k-i} \otimes \beta_i.$$

This is proven in Lemma 5.2 of Ravenel-Wilson [RW80] or in Adams [Ada95, Part II, Lemma 2.5]. The algebra structure on $K_*B^2\mathbf{Z}$ follows from the formal group law of K (recall that we write $K = K_2$) and is given by

$$K_*B^2\mathbf{Z} \cong K_*[\beta_{(i)} : i \geq 0]/(\beta_{(i)}^p - \beta_{(i-1)}),$$

where $\beta_{(i)} := \beta_{p^i}$ and $\beta_{(i)} = 0$ for $i < 0$. This is shown in Theorem 5.7 of [RW80]. It will be a common convention that $(i) = p^i$, and where circle products are involved, $x_{(k,l)}$ will mean $x_{(k)} \circ x_{(l)}$.

The space $B\mathbf{Z}/(p^j)$ is the fiber of the map $B^2\mathbf{Z} \to B^2\mathbf{Z}$ induced by multiplication by p^j. The K-homology of $B\mathbf{Z}/(p^j)$ is a sub-Hopf algebra of $K_*B^2\mathbf{Z}$ via $\delta_* = K_*(\delta)$. As a K_*-vector space $K_*B\mathbf{Z}/(p^j)$ is free on a_i, $0 \leq i < p^{2j}$, with $\delta_*(a_i) = \beta_i$, and as a Hopf algebra

$$K_*B\mathbf{Z}/(p^j) \cong P_{(2j)}(a_{(2j-1)}).$$

This is proven, again in Section 5 of Ravenel-Wilson [RW80], using the Gysin sequence for generalized homology.

In this subsection we describe $\mathrm{Tor}_{*,*}^{K_*B\mathbf{Z}/(p^j)}(K_*,K_*)$ as well as the homomorphism

$$\mathrm{Tor}_{*,*}^{K_*B\mathbf{Z}/(p^j)}(K_*,K_*) \to \mathrm{Tor}_{*,*}^{K_*B\mathbf{Z}/(p^{j+1})}(K_*,K_*)$$

induced by $\alpha : B\mathbf{Z}/(p^j) \to B\mathbf{Z}/(p^{j+1})$. We also take take care in computing the K-homology of the inclusion $\alpha : B\mathbf{Z}/(p^j) \to B\mathbf{Z}/(p^{j+1})$ induced by the inclusion $\mathbf{Z}/(p^j) \to \mathbf{Z}/(p^{j+1})$.

Lemma 6.1.1. *The map*

$$\alpha_* = K_*(\alpha) : K_*B\mathbf{Z}/(p^j) \to K_*B\mathbf{Z}/(p^{j+1}),$$

where $\alpha : B\mathbf{Z}/(p^j) \to B\mathbf{Z}/(p^{j+1})$ is induced by the inclusion $\alpha : \mathbf{Z}/(p^j) \to \mathbf{Z}/(p^{j+1})$, is given by

$$\alpha_*(a_{(2j-1)}) = a_{(2j+1)}^{p^2} = a_{(2j-1)}.$$

Proof. The diagram

$$\begin{tikzcd} B\mathbf{Z}/(p^j) \arrow[r, "\delta"] \arrow[d, "\alpha"'] & B^2\mathbf{Z} \arrow[d, equal] \\ B\mathbf{Z}/(p^{j+1}) \arrow[r, "\delta"] & B^2\mathbf{Z} \end{tikzcd}$$

commutes, so

$$\beta_{(2j-1)} = \delta_*(a_{(2j-1)}) = \delta_*(\alpha_*(a_{(2j-1)})),$$

and since $\beta_{(2j-1)} = \beta_{(2j+1)}^{p^2}$ and $\beta_{(2j+1)} = \delta_*(a_{(2j+1)})$ the result follows because as claimed earlier $K_*B\mathbf{Z}/(p^j)$ is a sub-Hopf algebra of $K_*B^2\mathbf{Z}$ via δ_*. $\square$

Lemma 6.1.2. *Since $K_*B\mathbf{Z}/(p^j) \cong P_{(2j)}(a_{(2j-1)})$ we get*

$$\mathrm{Tor}_{*,*}^{K_*B\mathbf{Z}/(p^j)}(K_*,K_*) \cong E(\sigma a_{(2j-1)}) \otimes \Gamma(\phi a_{(0)}),$$

where $\sigma a_{(2j-1)} = [a_{(2j-1)}]$ and $\phi a_{(0)} = [a_{(0)}^{p-1}|a_{(0)}]$. Here $a_{(0)}$ is $a_{(2j-1)}^{p^{2j-1}}$.

Proof. The first statement is a direct application of Lemma 4.4.3. $\square$

This lemma establishes the E^2-term of the spectral sequence

$$E^r_{*,*}(B\mathbf{Z}/(p^j)) \Rightarrow K_*B^2\mathbf{Z}/(p^j)$$

for $j \geq 1$. Next, we relate $E^2_{*,*}(B\mathbf{Z}/(p^j))$ to $E^2_{*,*}(B\mathbf{Z}/(p^{j+1}))$.

Lemma 6.1.3. *With the notation of the previous lemma, the homomorphism*

$$\alpha_* : E(\sigma a_{(2j-1)}) \otimes \Gamma(\phi a_{(0)}) \to E(\sigma a_{(2j+1)}) \otimes \Gamma(\phi a_{(0)})$$

induced by $\alpha : B\mathbf{Z}/(p^j) \to B\mathbf{Z}/(p^{j+1})$ *sends* $\sigma a_{(2j-1)}$ *to* 0 *and* $\phi a_{(0)}$ *to* $\phi a_{(0)}$.

Proof. The map α_* is accessible through direct computation. First, in the bar complex

$$\alpha_*[a_{(2j-1)}] = [a_{(2j+1)}^{p^2}] = -d_2[a_{(2j+1)}^{p^2-1}|a_{(2j+1)}] \sim 0,$$

which proves that $\alpha_*(\sigma a_{(2j-1)}) = 0$.

Second

$$\alpha_*[a_{(0)}^{p-1}|a_{(0)}] = \alpha_*[a_{(2j-1)}^{p^{2j-1}(p-1)}|a_{(2j-1)}^{p^{2j-1}}] = [a_{(2j+1)}^{p^{2j+1}(p-1)}|a_{(2j+1)}^{p^{2j+1}}] = [a_{(0)}^{p-1}|a_{(0)}],$$

so $\alpha_*(\phi a_{(0)}) = \phi a_{(0)}$. □

6.2 Frobenius and Verschiebung, properties of the circle product

We here recall some properties of the circle product. We need Frobenius reciprocity of the Frobenius and Verschiebung maps,

$$F(V(x) \circ y) = x \circ F(y),$$

and the result $V(a_{(i+1)}) = a_{(i)}$. These maps, F and V, are defined, and these properties are shown, in Section 7 of Ravenel-Wilson [RW80].

Lemma 6.2.1 (Lemma 11.2 [RW80])**.** *In* $K_*B^2\mathbf{Z}/(p^j)$,

$$a_{(i)} \circ a_{(k)} = -a_{(k)} \circ a_{(i)}, \tag{6.1}$$

$$a_{(i)} \circ a_{(i)} = 0, \tag{6.2}$$

$$a_{(i)} \circ a_{(k)} = 0 \quad \text{if } i < 2 \text{ and } k < 2(j-1) \tag{6.3}$$

$$= a_{(i-2)} \circ a_{(k+2)} \quad \text{if } 2 \leq i \text{ and } k < 2(j-1). \tag{6.4}$$

Proof. This is Lemma 11.2 in the same article [RW80]. □

This is enough to solve the multiplicative extension problems of $K_*B^2\mathbf{Z}/p$ ($j = 1$).

Lemma 6.2.2. *In* $K_*B^2\mathbf{Z}/p$ *we have*

$$a_{(0,1)}^p = -a_{(0,1)}$$

Proof. By direct computation using Frobenius reciprocity and Lemma 6.2.1, we get

$$a_{(0,1)}^p = F(a_{(0)} \circ a_{(1)}) = F(V(a_{(1)}) \circ a_{(1)}) = a_{(1)} \circ F(a_{(1)}) = -a_{(0,1)}.$$

□

6.3 The spectral sequence $E^r_{*,*}(\mathbf{Z}/(p^j)) \Rightarrow K_*B\mathbf{Z}/(p^j)$ for $j \geq 1$

We now proceed to determine the differential structure of the spectral sequence

$$\mathrm{Tor}^{K_*\mathbf{Z}/(p^j)}(K_*, K_*) \Rightarrow K_*B\mathbf{Z}/(p^j)$$

given knowledge of the abutment, which is essentially Section 8 of [RW80]. We also describe here the morphisms induced via naturality of the bar construction with respect to the standard inclusions $\alpha : \mathbf{Z}/(p^j) \to \mathbf{Z}/(p^{j+1})$.

Lemma 6.3.1. *Write $q = p^j$. $A = K_*\mathbf{Z}/(q)$ is the group ring $K_*[\mathbf{Z}/q]$, where we write $\mathbf{Z}/(q)$ as $\{e, g, \dots, g^{q-1}\}$ with multiplicative notation, and write $x = g - e$ in the group ring $K_*[\mathbf{Z}/q]$. Then*

$$\mathrm{Tor}^A_{*,*}(K_*, K_*) \cong E(\sigma x) \otimes \Gamma(\phi x^{p^{j-1}})$$

where $\sigma x = [x]$ and $\phi x^{p^{j-1}} = [x^{p^{j-1}(p-1)} | x^{p^{j-1}}]$.

Proof. The group ring $K_*[\mathbf{Z}/q]$ is a truncated polynomial algebra of height q on $x = g - e$ so the result follows from Lemma 4.4.3. $\square$

The differentials of

$$\mathrm{Tor}^{K_*\mathbf{Z}/(p^j)}(K_*, K_*) \Rightarrow K_*B\mathbf{Z}/(p^j)$$

are readily accessible:

Lemma 6.3.2. *In the spectral sequence $E^r_{*,*}(\mathbf{Z}/(p^j)) \Rightarrow K_*B\mathbf{Z}/(p^j)$ the first non-trivial differential is*

$$d^{2p^{2j}-1}(\gamma_{(2j)}\phi x^{p^{j-1}}) \doteq \sigma x,$$

so $E^{2p^{2j}}_{*,*} = E^\infty_{*,*}$ is the sub-Hopf algebra of $\Gamma(\phi x^{p^{j-1}})$ generated by $\gamma_i \phi x^{p^{j-1}}$ for $0 \leq i < p^{2j}$.

Proof. The K_*-dimension of the abutment $P_{(2j)}(a_{(2j-1)})$ is p^{2j} and so implies existence of non-trivial differentials. Lemma 5.5.2 (a) that determines their form is

$$d^{2p^k-1}\gamma_{p^k}\phi x^{p^{j-1}} \doteq \sigma x.$$

It then follows from Proposition 4.4.3 that for the smallest such k the dimension of $E^\infty = E^{2p^k}$ is p^k, so we must have $k = 2j$. $\square$

Next we describe representatives in the spectral sequence for elements in the abutment.

Lemma 6.3.3. *The element $\gamma_i \phi x^{p^{j-1}} \in E^{\infty}_{2i,*}(\mathbf{Z}/(p^j))$, $0 \leq i < p^{2j}$, represents $a_i \in P_{(2j)}(a_{(2j-1)}) \cong K_* B\mathbf{Z}/(p^j)$ modulo decomposables in E^{∞}.*

Proof. We compare with $E^r_{*,*}(S^1) \Rightarrow K_* \mathbf{C}P^{\infty}$. Since β_i is in bar filtration $2i$, the bar filtration of a_i cannot be less than a_i. If it were greater than $2i$, a_i would be represented by a nonzero class in $E^{\infty}_{s,t}$ for some $s > 2i$. By injectivity of the map of E^{∞}-terms, its image β_i would be represented by a nonsero class in bar filtration $s > 2i$, contradicting the fact that it is represented by $\gamma_i \beta$ in filtration $2i$. Hence a_i is represented in filtration $s = 2i$ by a class that maps to $\gamma_i \beta$. The only such class is $\gamma_i \phi x^{p^{j-1}}$. $\square$

Remark. Having chosen an imbedding $\mathbf{Z}/p^{\infty} \to S^1$, these identifications will be compatible for varying j because

$$\alpha : \mathbf{Z}/(p^j) \to \mathbf{Z}/(p^{j+1})$$

takes $\gamma_i \phi x^{p^{j-1}}$ to $\gamma_i \phi x^{p^j}$.

6.4 The spectral sequence $E^r_{*,*}(B\mathbf{Z}/(p)) \Rightarrow K_* B^2 \mathbf{Z}/(p)$

By Proposition 2.3 of Ravenel-Wilson [RW80] the map

$$E^1_{2,*}(\mathbf{Z}/p^j) \otimes K_* B\mathbf{Z}/p^j \to E^1_{2,*}(B\mathbf{Z}/p^j)$$

is given by

$$[x^{p^j-1}|x] \circ a_k = \sum_{r+s=k, 0<r,s<k} [x^{p^j-1} \circ a_r | x \circ a_s],$$

so we first compute $x \circ a_r$ and $x \circ a_s$ as a corollary of the following lemma.

Lemma 6.4.1. *The action of g^k on a_l under the circle product, for $0 \leq k < p^j$ and $0 \leq l < p^{2j}$, is given by $g^k \circ a_l = k^l a_l$.*

Proof. Write $\mathbf{C}P^{\infty} = K(\mathbf{Z}, 2)$ and denote by $k : \mathbf{C}P^{\infty} \to \mathbf{C}P^{\infty}$ the map induced by multiplication by $k \in \mathbf{Z}$. From the definition of the circle product it is clear that $k_*(a_l) = g^k \circ a_l$. Since

$$\begin{tikzcd}
\mathbf{C}P^{\infty} \arrow[r, "k"] \arrow[d, "\Delta^{(k)}"'] & \mathbf{C}P^{\infty} \arrow[d, equal] \\
\mathbf{C}P^{\infty} \times \cdots \times \mathbf{C}P^{\infty} \arrow[r, "\mu^{(k)}"'] & \mathbf{C}P^{\infty},
\end{tikzcd}$$

and $x \in K_*\mathbf{C}P^\infty$ is primitive, $k^*(x) = kx$, while $k_*(x^i) = k^i x^i$. Dually $\beta_i \mapsto k^i\beta_i$, and since

$$\begin{tikzcd} K_*\mathbf{C}P^\infty \arrow[r, "k_*"] & K_*\mathbf{C}P^\infty \\ B\mathbf{Z}/(p^j) \arrow[u, "\delta_*"] \arrow[r, "k_*"'] & B\mathbf{Z}/(p^j), \arrow[u, "\delta_*"'] \end{tikzcd}$$

commutes the result follows, as a_i it the unique element that maps to β_i under δ_*. $\square$

This allows us to compute the action of some power of x on an element a_l.

Lemma 6.4.2. *Said action is given by*

$$x^d \circ a_l = \begin{cases} d!a_l & l = d \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. By a direct computation

$$\begin{aligned} x^d \circ a_l &= \sum_{k=0}^{d} (-1)^k \binom{d}{k} g^k \circ a_l = \sum_{k=0}^{d} (-1)^k \binom{d}{k} k^l a_l \\ &= d! \left\{ {l \atop d} \right\} a_l = \begin{cases} d!a_l & l = d \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Here the *Stirling number of the second kind* $\left\{ {l \atop d} \right\} := \frac{1}{d!} \sum_{k=0}^{l} (-1)^k \binom{d}{k} k^l$, see [GKP94, Chapter 6.1, Table 264], is the number of ways to partition l elements into d sets. $\square$

Recall that

$$E^2_{*,*}(\mathbf{Z}/(p)) \cong E(\phi x) \otimes \Gamma(\phi x) \Rightarrow K_*B\mathbf{Z}/(p)$$

and that

$$E^2(B\mathbf{Z}/(p)) \cong E(\sigma a_{(1)}) \otimes \Gamma(\phi a_{(0)}) \Rightarrow K_*B^2\mathbf{Z}/(p).$$

Lemma 6.4.3. *For the map* $E^2_{*,*}(\mathbf{Z}/(p)) \otimes K_*B\mathbf{Z}/(p) \to E^2_{*,*}(B\mathbf{Z}/(p))$,

$$\phi x \circ a_{(1)} := [x^{p-1}|x] \circ a_{(1)} = [a_{(0)}^{p-1}|a_{(0)}] =: \phi a_{(0)}.$$

Proof. By a direct computation we get

$$[x^{p-1}|x] \circ a_{(1)} = \sum_{0<k,l<p,k+l=p} [x^{p-1} \circ a_k | x \circ a_l]$$
$$= \sum_{0<k,l<p,k+l=p} [(p-1)! \left\{ \begin{matrix} k \\ p-1 \end{matrix} \right\} a_k | a_l].$$

By the previous lemma this is equal to

$$= (p-1)![a_{p-1}|a_1]$$
$$= [a_{(0)}^{p-1}|a_{(0)}].$$

□

Since $\phi x \in E^\infty$ of $E^2_{*,*}(\mathbf{Z}/(p)) \cong E(\phi x) \otimes \Gamma(\phi x) \Rightarrow K_* B\mathbf{Z}/(p)$ represents $a_{(0)}$ modulo decomposables this lemma tells us that if the permanent cycle $\phi a_{(0)}$ survives to the E^∞-page, then $a_{(0,1)} = a_{(0)} \circ a_{(1)}$ is non-zero and $\phi a_{(0)} \in E^\infty$ of $E^2(B\mathbf{Z}/(p)) \cong E(\sigma a_{(1)}) \otimes \Gamma(\phi a_{(0)}) \Rightarrow K_* B^2\mathbf{Z}/(p)$ represents it modulo decomposables.

Theorem 6.4.4 (9.4(b), $q = 2$, $j = 1$)**.** *In the spectral sequence*

$$E^2(B\mathbf{Z}/(p)) \cong E(\sigma a_{(1)}) \otimes \Gamma(\phi a_{(0)}) \Rightarrow K_* B^2\mathbf{Z}/p$$

the first non-trivial differential is given by

$$d^{2p-1}\gamma_p \phi a_{(0)} \doteq \sigma a_{(1)}.$$

Proof. Recall that $\phi a_{(0)}$ is a permanent cycle representing $a_{(0,1)}$. If $\gamma_p \phi a_{(0)}$ supports a non-trivial differential the differential is as above, by part (a) of Lemma 5.5.2, because $\phi a_{(0)}$ is known to survive. Assume $\gamma_p \phi a_{(0)}$ is a permanent cycle that survives to an element $z \in K_* B^2\mathbf{Z}/(p)$. Since $p = 0 \in \mathbf{Z}/p$, $0 = [p] \circ x = FV(z) = F(a_{(0,1)})$, where $a_{(0,1)}$ is represented by $\phi a_{(0)}$, so

$$0 = FV(z) = F(a_{(0,1)}) = -a_{(0,1)} \neq 0$$

as is shown above, contradicting the assumption that $\gamma_p \phi_{(0)}$ survives. □

Lemma 6.4.5 (Lemma 9.7 [RW76], $q+1 = 2$)**.** *The class $\phi a_{(0)} \in E^2$ is a permanent cycle that survives to the E^∞-page, and modulo decomposables in E^∞ the class $\phi a_{(0)} \in E^\infty_{2,*}$ represents $a_{(0,1)}$.*

Proof. We know ϕx is a permanent cycle in $E^r(\mathbf{Z}/(p))$ representing $a_{(0)}$, and thus that $\phi x \circ a_{(1)} = \phi a_{(0)}$ represents $a_{(0,1)}$. Since ϕx is a permanent cycle and $d^r(\phi x \circ a_{(1)}) = d^r(\phi x) \circ a_{(1)} = 0$, $\phi a_{(0)}$ is a permanent cycle, and it survives to E^∞ for degree reasons. □

6.5 The spectral sequence $E^r_{*,*}(B\mathbf{Z}/(p^j)) \Rightarrow K_*B^2\mathbf{Z}/(p^j)$ for $j > 1$

Proposition 6.5.1 (Proposition 11.4 [RW80], α, $q = n = 2$)**.** *Let*

$$\alpha : B^2\mathbf{Z}/(p^j) \to B^2\mathbf{Z}/(p^{j+1}) \quad \textit{and} \quad \beta : B^2\mathbf{Z}/(p^j) \to B^2\mathbf{Z}/(p^{j+1})$$

be induced by the standard inclusion $\alpha\mathbf{Z}/(p^j) \to \mathbf{Z}/(p^{j+1})$ *and reduction* $\beta : \mathbf{Z}/(p^{j+1}) \to \mathbf{Z}/(p^j)$*, respectively. Then*

$$\alpha_*(a_{(i_1,i_2)}) = a_{(i_1,i_2+2)} \in K_*B^2\mathbf{Z}/(p^{j+1})$$

and

$$\beta_*(a_{(i_1+2,i_2+2)}) = a_{(i_1,i_2)} \in K_*B^2\mathbf{Z}/(p^j).$$

Lemma 6.5.2 (Theorem 11.1(b) [RW80])**.** *Any non-zero* $a_{(i_1,i_2)}$ *may be written on the form* $(-1)^{i_1}a_{(2k,2j-1)}$.

Remark. This result relies on Lemma 6.2.1. The proof is convoluted, but the bookkeeping is easier if one considers the relations of loc. cit. as transformations of the plane coordinates (i_1, i_2).

Proof. We first show that if $a_{(i_1,i_2)}$ is non-zero, then $i_1 \not\equiv i_2 \pmod 2$. We may take $i_1 < i_2$. If $i_1 \equiv i_2 \pmod 2$ then $i_1 = 2k + c$ and $i_2 = 2l + c$ with $0 \le c < 2$. Also $l = k + d$, $d > 0$, and for $2(k + d) + c < 2(j - 1)$, we get

$$a_{(2k+c,2(k+d)+c)} = -a_{(2(k+d)+c,2k+c)} = -a_{(2k+c,2(k+d)+c)} = 0,$$

which automatically vanishes in the case $k = 0$. For $2(k + d) + c = 2(j - 1) + c$ we get

$$a_{(2k+c,2(j-1)+c)} = -a_{(2(j-1)+c,2k+c)} = 0$$

because $2k + c < 2(j - 1)$. We have proved that if $a_{(i_1,i_2)}$ is non-zero, then $i_1 \not\equiv i_2 \pmod 2$.

Now if $a_{(i_1,i_2)}$ is non-zero and i_1 is odd, use $a_{(i_1,i_2)} = -a_{(i_2,i_1)}$. If $a_{(i_1,i_2)}$ is non-zero and i_1 is even with $i_1 \ge 2$ and $i_2 < 2(j - 1)$ apply $a_{(i_1,i_2)} = a_{(i_1-2,i_2+2)}$ until the second index is $2j - 1$. $\square$

Theorem 6.5.3 (Theorem 11.5 (c), $q = n = 2$, $i_1 = 2k$)**.** *Let* $0 \le k \le j - 1$. *Then modulo decomposables in* $E^\infty_{*,*}(B\mathbf{Z}/(p^j))$, $a_{(2k,2j-1)}$ *is represented by*

$$(-1)^k \gamma_{p^k}\phi a_{(0)}.$$

Proof. Note that $\alpha : B\mathbf{Z}/(p^{j-1}) \to B\mathbf{Z}/(p^j)$ induces a map

$$\alpha_* : \mathrm{Tor}_{*,*}^{K_*B\mathbf{Z}/(p^{j-1})}(K_*, K_*) \to \mathrm{Tor}_{*,*}^{K_*B\mathbf{Z}/(p^j)}(K_*, K_*).$$

For $k < j-1$ we prove the statement by induction. Assume that $a_{(2k,2(j-1)-1)}$ has been identified in $\mathrm{Tor}_{*,*}^{K_*\mathbf{Z}/(p^{j-1})}$ as $(-1)^k\gamma_{p^k}\phi a_{(0)}$. Applying α_* we get by Proposition 6.5.1

$$\alpha(a_{(2k,2(j-1)-1)}) = a_{(2k,2j-1)}$$

is represented by

$$\alpha_*((-1)^k\gamma_{p^k}\phi a_{(0)}) = (-1)^k\gamma_{p^k}\phi a_{(0)}$$

in $H_{*,*}K_*B\mathbf{Z}/(p^j)$ by naturality. In the base case, $j = 2$, we have already shown that $\phi a_{(0)}$ represents $a_{(0,1)}$ and so we have the result for $k < j-1$.

The case $k \geq j-1$ remains. Note that

$$V(a_{(2k,2j-1)}) = a_{(2k-1,2j-2)} = -a_{(2j-2,2k-1)} = -a_{(2j-2-2h,2k-1+2h)} = -a_{(2(k-1),2j-1)}$$

for $h = j-k$. For $k = j-1$ we thus see that

$$V^{j-1}(a_{(2(j-1),2j-1)}) = (-1)^{j-1}a_{(0,2j-1)},$$

which we have shown is represented by $\phi a_{(0)} = (-1)^{j-1}V^{j-1}(\gamma_{(j-1)}\phi a_{(0)})$. Hence the element $a_{(2(j-1),2j-1)}$ is represented by $\gamma_{(j-1)}\phi a_{(0)}$ modulo the kernel of V^{j-1}, the decomposables. Furthermore, $\gamma_{(j-1)}\phi a_{(0)}$ must be non-zero at the E^∞-page because $a_{(2(j-1),2j-1)}$ reduces under β to the non-trivial $a_{(2(j-2),2(j-1)-1)}$. $\square$

Corollary (Theorem 11.1(b)). *The $a_{(2k,2j-1)}$ with $0 \leq k \leq j-1$ are all non-zero and disctinct.*

Theorem 6.5.4 (Theorem 11.5(b)). *In $E^r(B\mathbf{Z}/p^j) \Rightarrow K_*B^2\mathbf{Z}/p^j$ the first non-trivial differential is given by*

$$d^{2p^j-1}\gamma_{(j)}\phi a_{(0)} \doteq \sigma a_{(2j-1)},$$

and so

$$E^{2p^j}_{*,*} \cong E^\infty_{*,*} \cong \bigotimes_{0\leq k\leq j-1} P_p(\gamma_{(k)}a_{(0)}).$$

Proof. We first show that $\gamma_{(j)}\phi a_{(0)}$ does not survive. Assume first that $\gamma_{(j)}\phi a_{(0)}$ survives to represent an element $z \in K_*B^2\mathbf{Z}/p^j$. The map $p^j : B^2\mathbf{Z}/p^j \to B^2\mathbf{Z}/p^j$ is null-homotopic and so

$$0 = p^j_*(z) = p^{j-1}_*(V(z))^p,$$

where $V(z)$, by inspection of the spectral sequence, is represented by $a_{(2(j-1),2j-1)}$ in the abutment, so

$$p_*^{j-1}(V(z))^p = p_*^{j-1}(a_{(2(j-1),2j-1)}) = a_{(0,2j-1)}$$

which is already known to be non-zero and represented by $\phi a_{(0)}$, contradicting the assumption that it does. It is the differential of lowest degree and determines the differentials of the spectral sequence. □

We are now in a position to solve all multiplicative extension problems in the spectral sequence

$$\mathrm{Tor}_{*,*}^{K_*B\mathbf{Z}/(p^j)}(K_*, K_*) \Rightarrow K_*B^2\mathbf{Z}/(p^j).$$

The problem is that $(\gamma_{(k)}\phi a_{(0)})^p = 0$ in the E^∞-page, but the element $(a_{(2k,2j-1)})^p$ that it represents is only zero modulo filtration.

Theorem 6.5.5. *As a Hopf algebra*

$$K_*B^2\mathbf{Z}/(p^j) \cong \bigotimes_{k=0}^{j-1} K_*[a_{(2k,2j-1)}]/(a^p_{(2k,2j-1)} + a_{(2k,2j-1)}).$$

Proof. Recall that $\gamma_{(k)}\phi a_{(0)}$ represents $a_{(2k,2j-1)} = a_{(2k)} \circ a_{(2j-1)}$. We have that $a_{(2k)} = V(a_{(2k+1)})$ ([RW80], as mentioned above). Then by the Frobenius reciprocity relation of Section 6.2

$$\begin{aligned}(a_{(2k)} \circ a_{(2j-1)})^{*p} &= F(V(a_{(2k+1)}) \circ a_{(2j-1)}) \\ &= a_{(2k+1)} \circ F(a_{(2j-1)}) \\ &= a_{(2k+1)} \circ a_{(2j-2)},\end{aligned}$$

which for some h, by part (6.4) of Lemma 6.2.1, is equal to

$$a_{(2k+1+2h)} \circ a_{(2j-2-2h)}.$$

For $h = j - k - 1$ we see that

$$\begin{aligned}a_{(2k+1+2h)} \circ a_{(2j-2-2h)} &= a_{(2k+1+2(j-k-1))} \circ a_{(2j-2-2(j-k-1))} \\ &= a_{(2j-1)} \circ a_{(2k)} \\ &= -a_{(2k)} \circ a_{(2j-1)}.\end{aligned}$$

This solves all extension problems and the result follows. □

7 Ring spectrum maps from $\Sigma^\infty B^3\mathbf{Z}_+$ to K_2

From Theorem 6.5.5 we know that

$$K_*B^2\mathbf{Z}/(p^j) \cong \bigotimes_{k=0}^{j-1} K_*[a_{(2k,2j-1)}]/(a^p_{(2k,2j-1)} + a_{(2k,2j-1)}).$$

From essentially the same discussion as in [RW80, Section 12], $K_*B^2\mathbf{Z}/(p^j)$ is a sub-Hopf algebra of $K_*B^3\mathbf{Z}$ and

$$K_*B^3\mathbf{Z} \cong \operatorname{colim}_j B^2\mathbf{Z}/(p^j) \cong \bigotimes_{k\geq 0} K_*[b_{(2k,1)}]/(b^p_{(2k,1)} + b_{(2k,1)}).$$

Write $b_{(2k,1)}$ for the image of $a_{(2k,2j-1)}$ in $K_*B^3\mathbf{Z}$. The current chapter serves as a proof of the following.

Remark. Note that the $b_{(2k,1)}$ may be taken to lie in degree 0.

Proposition 7.0.6. *The ring spectrum maps $\Sigma^\infty B^3\mathbf{Z}_+ \to K_2$ are in one-to-one correspondence with characters*

$$K_*B^3\mathbf{Z} \cong (K_*B^3\mathbf{Z})^* \to \mathbf{F}_p^*.$$

These are determined by the images of $b_{(2k,1)}$ in $\mathbf{F}_p^*$ for $k = 0, 1, 2, \ldots$.

Proof. Lemma 7.1.1 shows that ring spectrum maps $\Sigma^\infty B^3\mathbf{Z} \to K_2$ correspond to group-like elements in $K^*B^3\mathbf{Z} \cong (K_*B^3\mathbf{Z})^*$. Lemma 7.2.2 determines the group-like elements in terms of multiplicative functionals on each tensorand, and Lemma 7.2.3 shows that they are determined by an element in $\mathbf{F}_p^*$. $\square$

Remark. It follows from Lemma 7.2.2 and Lemma 7.2.3 that there does not exist a non-trivial ring spectrum map $B^3\mathbf{Z}_+ \to K(2)$.

7.1 Cohomological detection of ring spectrum maps

In this section we prove an algebraic criterion for identifying when a cohomology class is represented by a ring spectrum map. Recall that if G and E are ring spectra with E having a Künneth theorem, the E-cohomology of G has the structure of a Hopf algebra. We identify the ring spectrum maps $G \to E$ (i.e., ring maps in the stable homotopy category of spectra) in terms of this Hopf algebra structure.

Lemma 7.1.1. *Let G and E be as above. Then group-like elements in $x \in E^0G$ with augmentation $\epsilon(x) = 1 \in E^0$ are exactly the ring spectrum maps $G \to E$.*

Proof. Write μ_G, η_G and μ_E, η_E for the multiplication and unit in G and E respectively, write

$$K : E^*G \otimes_{E^*} E^*G \to E^*(G \wedge G)$$

for the Künneth isomorphism, and denote by Δ the coproduct on E^*G given by $K^{-1} \circ \mu_G^*$.

Let $f : G \to E \in E^*G$ be group-like. Then $f \circ \mu_G \in E^*(G \wedge G)$ corresponds to $f \otimes f \in E^*G \otimes_{E^*} E^*G$ under the Künneth isomorphism K. Under the factorization

$$\begin{tikzcd}
E^*G \otimes_{E^*} E^*G \arrow[r, "g\otimes h \mapsto g \wedge h"] \arrow[d, equal] & (E \wedge E)^*(G \wedge G) \arrow[d, "\mu_E \circ (-)"] \\
E^*G \otimes_{E^*} E^*G \arrow[r, "K"'] & E^*(G \wedge G)
\end{tikzcd}$$

we see that

$$f \otimes f \mapsto \mu_E \circ f \wedge f = f \circ \mu_G,$$

and thus f is a ring spectrum map.

Let now $f : G \to E$ be a ring spectrum map. Then $\mu_E \circ (f \wedge f) = f \circ \mu_G$. The element $f \otimes f \in E^*G \otimes_{E^*} E^*G$ maps to $\mu_E \circ (f \wedge f)$ under the factorization above. Since K is an isomorphism, the map

$$E^*G \otimes_{E^*} E^*G \to (E \wedge E)^*(G \wedge G)$$

commutes, and

$$K(f \otimes f) = \mu_E \circ (f \wedge f) = (f \wedge f) \circ \mu_G.$$

Applying K^{-1} to both sides we get

$$f \otimes f = K^{-1}((f \wedge f) \circ \mu_G) = \Delta(f),$$

i.e., f is group-like.

It follows from the definitions that an element $f \in E^0(G)$ satisfies $\epsilon(f) = 1$ if and only if $f \circ \eta_G = \eta_E$. $\square$

7.2 Group-like elements in K_2-cohomology of $B^3\mathbf{Z}$

Lemma 7.2.1. *Let $R = K[x]/(x^p + x)$ with multiplication μ, and let R^* be the dual coalgebra of R. Write $K\{c_0, \ldots, c_{p-1}\}$ with c_i dual to x^i, and let $\psi = \mu^*$ be the coproduct on R^*. Then for any c_k*

$$\psi(c_k) = \sum_{i+j=k} c_i \otimes c_j - \sum_{i+j=p+k-1} c_i \otimes c_j.$$

Proof. This follows from knowledge of the multiplication μ, and the fact that ψ is dual to μ. □

Lemma 7.2.2. *If $f = a_0c_0 + \cdots + a_{p-1}c_{p-1}$ is a non-zero group-like element of R^*, then $a_0 = 1$, $a_1^k = a_k$, and $a_1^{p-1} = -1$.*

Remark. This means that a group-like element f is determined by $a_1 \in K$.

Proof. If $f = a_0c_0 + \cdots + a_{p-1}c_{p-1}$ and $\psi(f) = f \otimes f$, i.e., f is group-like, we must have

$$\begin{aligned} f \otimes f &= \sum_{k=0}^{2p-2} \sum_{i+j=k} a_i a_j c_i \otimes c_j \\ &= \sum_{k=0}^{p-1} a_k \psi(c_k) \\ &= \sum_{k=0}^{p-1} a_k \sum_{i+j=k} c_i \otimes c_j - \sum_{k=0}^{p-1} a_k \sum_{i+j=p+k-1} c_i \otimes c_j. \end{aligned}$$

Comparing coefficients we find that

$$a_i a_j = \begin{cases} a_{i+j} & i+j < p \\ -a_{i+j-p+1} & i+j \geq p \end{cases}$$

From the relation $a_0 a_i = a_i$ we see that $a_0 = 1$. From the relation $a_1 a_i = a_{1+i}$ for $1+i < p$ we see inductively that $a_1^k = a_k$, $k = 1, \ldots, p-1$. Because $a_1^p = a_1 a_{p-1} = -a_1$, we see that $a_1^{p-1} = -1$. □

Lemma 7.2.3. *Let $K = \mathbf{F}_{p^2} = \mathbf{F}_p + \mathbf{F}_p\sqrt{d}$ with d a non-quadratic residue modulo an odd p. Then $(a\sqrt{d})^{p-1}$ with $a \in \mathbf{F}_p^*$ yield all $p-1$ solutions to the degree $p-1$ polynomial $x^{p-1} + 1$.*

Proof. Let $(d|p) \equiv d^{\frac{p-1}{2}} \pmod{p}$ be the Legendre symbol. It is -1 if $d \neq 0$ is a non-quadratic residue modulo p. Note that $p-1$ is even since p is odd. We compute:

$$(a\sqrt{d})^{p-1} = a^{p-1}d^{\frac{p-1}{2}} \equiv (d|p) = -1,$$

since d is a quadratic non-residue modulo p. $\square$

A Smallness

This appendix provides notions of smallness needed for the definition of a cofibrantly generated model category category. The small object argument is useful for constructing functorial factorizations for cofibrantly generated model categories. Grothendieck categories deal with set theoretical problems that arise when taking nerves of big categories.

A.1 Ordinal and cardinal numbers

For more details on ordinals and cardinals, see [Kun11].

Definition A.1.1. A linear order on a set T is called *well-ordered* if every non-empty set of elements $U \subset T$ has a smallest element.

Lemma A.1.2. *If T and T' are well-ordered sets that are isomorphic as well-orderings, then the isomorphism is unique.*

Proof. See Kunen [Kun11, Lemma 6.2]. □

Definition A.1.3. A set α is *transitive* if each of its elements are also subsets of α. The set α is an *ordinal* if it is a transitive set well-ordered with respect to inclusion.

Lemma A.1.4. *If α, β are ordinals, then either $\alpha = \beta$, $\alpha < \beta$ or $\beta < \alpha$. In other words, the class of ordinals is well-ordered with respect to inclusion.*

Proof. This is [Kun11, Theorem 7.3 (1)]. □

Lemma A.1.5. *Let A bet a set. Then there is a unique ordinal $|A|$ in bijection to it. Call $|A|$ the* cardinality *of A. If α is an ordinal then $|\alpha| = \alpha$.*

Proof. By the axiom of choice, the set A has a well-ordering. Then the lemma follows from [Kun11, Theorem 7.6]. □

Definition A.1.6. Denote by $\alpha + 1$ the minimal ordinal that dominates α.

A.2 Transfinite compositions

The current discussion essentially follows [Hov99, Chapter 2, Section 1.1]. Throughout the following, let $\mathscr{C}$ be a cocomplete category.

Definition A.2.1. Let λ be an ordinal number. A λ-sequence in $\mathscr{C}$ is a cocontinuous λ-shaped diagram

$$X : \lambda \to C,$$

often written

$$X_0 \to X_1 \to \cdots \to X_\beta \to \cdots .$$

Since X preserves colimits, if $\gamma < \lambda$ is a limit ordinal the induced morphism

$$\operatorname{colim}_{\beta<\gamma} X_\beta \to X_\gamma$$

is an isomorphism. Also note that we get a map

$$X_0 \to \operatorname{colim}_{\beta<\gamma} X_\gamma.$$

The map $X_0 \to \operatorname{colim}_{\beta<\lambda} X_\beta$ is referred to as the *composition* of the λ-sequence. The composition is not unique unless λ is a limit ordinal (according to Hovey it is only unique *up to isomorphism under* X.

If D is a collection of morphisms of $\mathscr{C}$ such that every map $X_\beta \to X_{\beta+1}$ is in D whenever $\beta + 1 < \lambda$, we refer to the composition $X_0 \to \operatorname{colim}_{\beta<\lambda} X_\beta$ as a *transfinite composition* of maps of D.[1]

Definition A.2.2. Let κ be a cardinal. A limit ordinal λ is *κ-filtered* if for any subset $S \subseteq \lambda$ with cardinality $|S| \leq \kappa$, the colimit of S is bounded by λ.

A subset S of an ordinal λ is *cofinal* in λ if every element of λ is dominated by some element of S. The *cofinality* of λ is the smallest ordinal number isomorphic to a cofinal subset of α. An ordinal λ is κ-filtered if and only if the cofinality of λ is greater than κ.

We want a "smallness" criterion on objects of $\mathscr{C}$, ensuring some control over morphisms to colimits over "big enough" λ-sequences. More precisely we would like the map $A \to \operatorname{colim}_{\beta<\lambda} X_\beta$ to factor through some stage of

$$X_0 \to \cdots \to X_\beta \to \cdots$$

[1] This is related to the Transfinite Induction Theorem, stating that if $\mathscr{C}$ is a class of ordinals containing 0, with $\alpha + 1 \in \mathscr{C}$ for all $\alpha \in \mathscr{C}$, and any limit ordinal α is in $\mathscr{C}$ whenever all bounded ordinals $\beta < \alpha$ are in $\mathscr{C}$, then $\mathscr{C}$ is the class of all ordinals.

in the composition $X_0 \to \operatorname{colim}_{\beta<\lambda} X_\beta$, that is, via one of the morphisms

$$X_\beta \to \operatorname{colim}_{\beta<\lambda} X_\beta.$$

Let A be an object of $\mathscr{C}$. Observe that the set $\operatorname{colim}_{\beta<\lambda} \mathscr{C}(A, X_\beta)$ has a presentation

$$\operatorname{colim}_{\beta<\lambda} C(A, X_\beta) = \coprod_{\beta<\lambda} C(A, X_\beta)/\sim$$

where

$$f_\beta : A \to X_\beta \sim f_{\beta'} : A \to X_{\beta'}$$

if and only if the composition of $f_\beta : A \to X_\beta$ with $X_\beta \to X_{\beta'}$ equals $f_{\beta'} : A \to X_{\beta'}$ whenever $\beta < \beta'$.

From this discussion, we see that $\operatorname{colim}_{\beta<\lambda} C(A, X_\beta)$ is the set of sequences of morphisms compatible with and factoring through some stage of the λ-sequence. Now, the family or morphisms $X_\beta \to \operatorname{colim}_{\beta<\lambda} X_\beta$ induces a morphism

$$\operatorname{colim}_{\beta<\lambda} C(A, X_\beta) \to C(A, \operatorname{colim}_{\beta<\lambda} X_\beta).$$

We see that if the corepresented functor $C(A, -)$ preserves colimits, the map $f : A \to \operatorname{colim}_{\beta<\lambda} X_\beta$ is the final morphism of a sequence of morphisms factoring through some stage of the composition as required.

Definition A.2.3 (κ-small objects)**.** Let still $\mathscr{C}$ be a cocomplete category, D be a collection of morphisms of $\mathscr{C}$, let A be an object of $\mathscr{C}$ and κ a cardinal. Then A is *κ-small relative to D* if for all κ-filtered ordinals λ and all λ-sequences $X : \lambda \to C$ with maps $X_\beta \to X_{\beta+1}$ in D for $\beta + 1 < \lambda$, the map of sets

$$\operatorname{colim}_{\beta<\lambda} C(A, X_\beta) \to C(A, \operatorname{colim}_{\beta<\lambda} X_\beta)$$

is a bijection.

The object A is said to be *small relative to* D if it is κ-small relative to D for some κ. It is *small* if it is small relative to $D =$ all morphisms of $\mathscr{C}$.

If $\kappa < \kappa'$ and A is κ-small relative to D, then A is κ'-small relative to D, since any κ'-filtered ordinal is κ-filtered.

Definition A.2.4 (Finitude)**.** Let still D be a collection of morphisms of $\mathscr{C}$. An object A of $\mathscr{C}$ is *finite* or *compact relative to* D if A is κ-small relative to D for a finite ordinal κ. We say that A is finite if it is finite relative to $\mathscr{C}$ itself. In this case, maps from A commute with colimits of arbitrary λ-sequences, where λ is a limit ordinal.

Example A.2.5. Every set is small. Indeed, a set A is $|A|$-small. Suppose an ordinal λ is $|A|$-filtered, and let X be a λ-sequences of sets. Given a map

$$f : A \to \operatorname{colim}_{\beta<\lambda} X_\beta,$$

there is for each $a \in A$ an index β_a with $f(a) \in X_{\beta_a}$. Then f will factor through X_γ, where γ is the supremum of the β_a. Similarly, if two maps $A \to X_\alpha$ and $A \to X_\beta$ are equal in the colimit, then they are equal in some stage of the composition. Note that a set is finite in $(Sets)$ if and only if it is a finite set.

Example A.2.6. If R is a ring, every R-module is small. Let $\mathscr{M}_R$ be the category of (left) R-modules, and suppose A is an R-module. Let $\kappa = |A|(|A| + |R|)$. If λ is κ-filtered and X is a λ-sequence of R-modules, then by the last example the map

$$\operatorname{colim} \mathscr{M}_R(A, X_\beta) \to \mathscr{M}_R(A, \operatorname{colim} X_\beta)$$

is injective, and any map $f : A \to \operatorname{colim} X_\beta$ factors as a map of sets $g : A \to X_\beta$ for some $\beta < \lambda$. For each pair $(x, y) \in A \times A$ there is a $\beta_{(x,y)}$ such that $g(x + y) = g(x) + g(y)$ in $X_{\beta_{(x,y)}}$. Similarly for $(r, x) \in R \times A$ there is a $\beta_{(x,y)}$ with $g(rx) = rg(x)$ in $X_{(r,x)}$. If γ is the supremum of these ordinals, then $\gamma < \lambda$ and the map g defines a factorization of f through an R-module map $A \to X_\gamma$. Note that finitely generated R-modules are finite.

A.3 The small object argument

The small object argument in a category enables the construction of a functorial factorization of a morphism $f : X \to Y$ into a composition $\delta(f) \circ \gamma(f)$ where $\gamma(f)$ is a generalized relative cell complex (a cofibration) and $\delta(f)$ a fibration. This construction is functorial (in the arrow category) as required by Hovey [Hov99], and the argument is useful for constructing functorial factorizations given a set of generating cofibrations for a model structure.

Definition A.3.1. Let I be a set of maps in a cocomplete category $\mathscr{C}$. A map $f : X \to Y$ is a *relative I-cell complex* if it is a transfinite composition of pushouts of elements of I, i.e., there is an ordinal λ, and a sequence $f_i : \lambda \to C$ such that f is the composition of the f_i (in the sense of [Hov99]), and for each β with $\beta + 1 < \lambda$ there is a pushout square

$$\begin{tikzcd} C_\beta \arrow[r] \arrow[d, "g_\beta"'] & X_\beta \arrow[d, "f_\beta"] \\ D_\beta \arrow[r] & X_{\beta+1} \end{tikzcd}$$

with $g_\beta \in I$. An object $X \in C$ is an *I-cell complex* if the morphism from the initial object is a relative I-cell complex.

Definition A.3.2. A map in $\mathscr{C}$ is *I-injective* if it has the left lifting properties with respect to maps in I. It is *I-projective* if it has the right lifting property with respect to I. A map is called an *I-cofibration* if it has the left lifting property with respect to all I-injective maps, and an *I-fibration* if it has the right lifting property with respect to all I-projective maps. The collection of I-injective maps, I-projective maps, I-cofibrations and I-fibrations, are denoted by I-inj, I-proj, I-cofib, and I-fib, respectively.

Theorem A.3.3 (The small object argument)**.** *Let $\mathscr{C}$ be a cocomplete category and I a set of maps in I. Suppose the domains of I are small relative to I-cell. Then there is a functorial factorization (γ, δ) on $\mathscr{C}$ such that for all $f \in C$, $\gamma(f)$ is an I-cell complex and the map $\delta(f)$ is in I-inj.*

In other words, we may construct functorial factorizations.

A.4 Grothendieck universes

Without care construction of the K-theory spectrum of a commutative symmetric ring spectrum involves taking the nerve of a category whose objects form a proper class. We bypass this using the theory of universes. By picking two universes U, V with $U \in V$ we may form the category of U-spectra. The category of modules over a U-ring spectrum is a U-small category, and the nerve of any subcategory is a V-small simplicial set, the geometric realization of which is a V-space. Another approach to nerves is to replace our category with a small category equivalent to it.

Definition A.4.1. Let U be a set. Then U is a *Grothendieck universe* if the following holds.

1. If $x \in U$ and $y \in x$, then $y \in U$.
2. If $x, y \in U$, then $\{x, y\} \in U$.
3. If $x \in U$, then its power-set $P(x) \in U$.
4. If $\{x_i\}, i \in I$ is a family of sets $x_i \in U$ with $I \in U$, then $\bigcup_{i\in I} x_i \in U$.

B Topics on categories and groups

B.1 Nerves and the classifying space of a category

For nerves of categories outside considerations of universes, see [ML98, Section 2, Chapter 12]. Let U and V with $U \in V$. Given a locally U-small, V-small category $\mathscr{C}$, the *nerve* of $\mathscr{C}$ is a V-simplicial set formed by considering $\Delta[n]$ a category letting $N_n\mathscr{C} = \hom([n], \mathscr{C})$, the V-small set of functors from $[n]$ to $\mathscr{C}$, i.e., strings $(g_1, \ldots, g_n)$ of n composable morphisms. The face maps are determined by composition and inserting identities as follows. The face maps $d_i : N.\mathscr{C}_n \to N.\mathscr{C}_{n-1}$ given by

$$\begin{aligned} d_i(g_1, \ldots, g_n) &= (g_2, \ldots, g_n) && (i = 0) \\ &= (g_1, \ldots, g_{n-1}) && (i = n) \\ &= (g_1, \ldots, g_{i+1} \circ g_i, \ldots, g_n) && (0 < i < n), \end{aligned}$$

and degeneracy maps $s_i : N.\mathscr{C}_n \to N.\mathscr{C}_{n-1}$ by

$$\begin{aligned} s_i(g_1, \ldots, g_n) &= (1, g_1, \ldots, g_n) && (i = 0) \\ &= (g_1, \ldots, g_n, 1) && (i = n) \\ &= (g_1, \ldots, g_{i-1}, 1, g_i, \ldots, g_n), && (0 < i < n) \end{aligned}$$

where 1 is the identity of the i-th object in the sequence.

Given a V-simplicial set we may define a geometric realization in the usual way by specifying $|\Delta[n]|$ to be the geometric n-simplex considered as a V-space (i.e., with a V-small underlying set), and forcing $|\cdot|$ to commute with colimits. The V-space $|N.\mathscr{C}|$ is called the *classifying space* of $\mathscr{C}$ and is sometimes written $B\mathscr{C}$.

B.2 Closed symmetric monoidal categories and Day convolution

A monoidal category is a category is one with enough structure to define associative monoid objects. A monoidal category is symmetric if it has a binatural twist isomorphism $X \otimes Y \cong Y \otimes X$. This allows defining commutative monoid objects in the category. A monoidal category is closed if there is an internal $\hom$-object X^Y that is left-adjoint to the tensor product in the sense that we're given a tri-natural isomorphism

$$\hom(X \otimes Y, Z) \cong \hom(X, Z^Y).$$

For details see Mac Lane [ML98, Chapter VII and Chapter XI].

If I is a small symmetric monoidal category, and $\mathscr{C}$ is a cocomplete category with tensor product $\otimes$, then Day [Day70] proves that the functor category $\mathscr{C}^I$ inherits a symmetric monoidal structure. For a discussion see [SS12, Section 2], [DGM13, Section 1.2.3] or [Day70]. The monoidal product is given by *Day convolution*, and may be computed from the coend formula

$$(A \otimes A')(k) = \int^{(i,j) \in I \times I} A(i) \otimes A(j) \times I(i \otimes j, k).$$

For coends see [ML98, Chapter IX]. If $- \otimes - \downarrow k$ is the category of triples (i, j, v) with v a structure map $i \otimes j \to k$, and morphisms of triples $(i, j, v) \to (i', j', v')$ are pairs of morphism $f_1 : i \to i'$ and $f_2 : j \to j'$ such that $f_1 \otimes f_2$ is compatible with the structure maps, the coend formula takes $A \otimes A'(k)$ the following expression:

$$A \otimes A'(k) = \operatorname{colim}_{(i,j,v) \in - \wedge - \downarrow k} A(i) \wedge A(j) \times I(i \otimes j, k).$$

This has the following immediate consequence.

Lemma B.2.1. *A map $A' \wedge A'' \to A$ out of a corresponds uniquely to a family of maps*

$$A'(i) \wedge A''(j) \to A(i \otimes j)$$

functorial in i and j.

B.3 The bar construction in a monoidal category

Let $\mathscr{C}$ be a (not necessarily symmetric) monoidal category with product $\otimes$ and unit 1, and let P be a monoid in $\mathscr{C}$. If M is a right P-module, and N is a right P-module, then we may form the *two-sided bar construction* $B^{\otimes}_{\cdot}(M, P, N)$. It is a simplicial object in $\mathscr{C}$ with

$$B^{\otimes}_k(M, P, N) = M \otimes \underbrace{P \otimes \cdots \otimes P}_{k} \otimes N.$$

The face maps are generated by the module structure $M \otimes P \to M$, $P \otimes N \to N$ and multiplication $P \otimes P \to P$, and degeneracy maps given by using the unit map $1 \to P$,

$$M \otimes P \otimes \cdots \otimes 1 \otimes \cdots \otimes P \otimes N \to M \otimes P \otimes \cdots \otimes P \otimes \cdots \otimes P \otimes N$$

in the obvious way. If P has an augmentation $P \to 1$ we write $B^{\otimes}_{\cdot}P = B^{\otimes}_{\cdot}(1, P, 1)$ for the *bar construction on* P.

B.4 A model for the classifying space of an (abelian) topological group

Let G be a topological group. We here present a model for the classifying space BG of G[1] as the geometric realization of (a model for) the nerve of the group, and give references for the results we shall need in the sequel.

Definition B.4.1. Let G be a topological abelian group. Then $BG = |B.G|$, the geometric realization of the bar construction on G in the category of sets, is the *classifying space of* G.

Remark. A useful fact is that the space ΩBG of loops in BG is homotopy equivalent to G, so BG is a delooping of G. The homotopy equivalence is the adjoint of the inclusion of $G \wedge S^1$ as the 1-simplexes of BG.

Lemma B.4.2. *Let G be a topological group where G has the basepoint e. Then classifying space BG is a topological group with a single n-cell for each non-degenerate n-vertex in $N.G_n$. Further the quotient of its s-skeleton BG_s by the $(s-1)$-skeleton, BG_s/BG_{s-1}, is homeomorphic to $G^{\wedge s} \wedge S^s$.*

Proof. We now give an explicit presentation of $BG = |NG|$. Write Δ^n for the standard topological n-simplex. Denote by $d^i : \Delta^{n-1} \to \Delta^n$ the standard inclusion of Δ^{n-1} as the

[1]Typically the classifying space BG is defined as the classifying space of principal G-bundles, i.e., a principal G-bundle P over a space X should be the pullback of a universal principal G-bundle EG over BG along a map $X \to BG$ well-defined up to homotopy. For more on this perspective on classifying spaces see Switzer [Swi75] or Husem̈oller [Hus94]

i-th face of Δ^n, and by also $s^i : \Delta^{n+1} \to \Delta^n$ for the standard collapse of Δ^{n+1} onto its i-th face $d^i(\Delta^n) \subset \Delta^n$. Then

$$|NG| = \left(\coprod_{n \geq 0} NG_n \times \Delta^n \right) / \sim$$

where

$$(d_i(g_1, \ldots, g_n), x) \in NG_{n-1} \times \Delta^{n-1} \sim ((g_1, \ldots, g_n), d^i(x)) \in NG_n \times \Delta^n$$

and

$$(s_i(g_1, \ldots, g_n), x) \in NG_{n+1} \times \Delta^{n+1} \sim ((g_1, \ldots, g_n), s^i(x)) \in NG_n \times \Delta^n.$$

Denote by BG_s the s-skeleton

$$BG_s = \bigcup_{k \leq s} \{[(g, x)] : g \in NG_k,\ x \in \Delta^k\}.$$

The first relation identifies the $(s-1)$-skeleton as a subspace of the s-skeleton, while the second relation collapses the $(n+1)$-simplices paired with the different degeneracies $s_i(g)$ of an n-simplex $g = (g_1, \ldots, g_n)$.

In the quotient BG_s/BG_{s-1} of the s-skeleton by the $(s-1)$-skeleton this tells us we have collapsed the boundary of any s-cell to a single point, while any (g, x) with g degenerate (that is, $g = (g_1, \ldots, g_n, x)$ with some $g_i = e$,) or x on some boundary $d^i(\Delta^{n-1}) \subset \Delta^n$, is collapsed to that same point $*$. Thus

$$BG_s/BG_{s-1} \cong \left(G^{\times s} \times \Delta^s\right) / \{(g_1, \ldots, g_n, x) \text{ with some } g_i = e \sim *\} \cong G^{\wedge s} \wedge S^s,$$

and we are done. $\square$

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